1-1 \textbf{Reteaching}
Nets and Drawings for Visualizing Geometry

A \textit{net} is a two-dimensional flat diagram that represents a three-dimensional figure. It shows all of the shapes that make up the faces of a solid.

Stepping through the process of building a three-dimensional figure from a net will help you improve your ability to visualize the process. Here are the steps you would take to build a square pyramid.

\textbf{Step 1} \hspace{1cm} \textbf{Step 2} \hspace{1cm} \textbf{Step 3}
Start with the net. \hspace{1cm} Fold up on the dotted lines. \hspace{1cm} Tape the adjacent triangle sides together.

Here are other examples of nets that also fold up into a square pyramid.

\textbf{Problem}
How can you be sure that none of the nets shown above are the same?
Make sure you cannot rotate or flip any net and place it on top of any other net.

\textbf{Exercise}
1. What is a possible net for the figure shown at the right? A

A. \hspace{1cm} B. \hspace{1cm} C.
An *isometric drawing* is a corner-view drawing of a three-dimensional figure. It shows the top, front, and side views.

To make an isometric drawing, you can start by visualizing picking up a block structure and turning it so that you are looking directly at one face. Draw the edges of that face. Then visualize and draw the two other faces.

Follow the steps below to make an isometric drawing of the block structure at the right.

**Step 1**
Draw the edges that surround the front face.

**Step 2**
Start at a vertex of the front face and draw an edge for the right side view. Edges only occur at bends and folds.

**Step 3**
Draw the back and top edges to connect the figure. Draw all parallel edges.

**Problem**
What is an isometric drawing of the block structure at the right?

Isometric drawing:

**Exercises**

2. Draw a possible net for this figure.

3. Make the isometric drawing for this structure.
Review these important geometric terms.

<table>
<thead>
<tr>
<th>Term</th>
<th>Examples of Labels</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>Italicized capital letter: D</td>
<td>![Point Diagram]</td>
</tr>
</tbody>
</table>
| Line       | Two capital letters with a line drawn over them: \( \overline{AB} \) or \( \overline{BA} \)  
One italicized lowercase letter: \( m \) | ![Line Diagram] |
| Line Segment| Two capital letters (called endpoints) with a segment drawn over them: \( \overline{AB} \) or \( \overline{BA} \) | ![Line Segment Diagram] |
| Ray        | Two capital letters with a ray symbol drawn over them: \( \overline{AB} \) | ![Ray Diagram] |
| Plane      | Three capital letters: \( ABF, AFB, BAF, BFA, FAB, \) or \( FBA \)  
One italicized capital letter: \( W \) | ![Plane Diagram] |

**Remember:**

1. When you name a ray, an arrowhead is not drawn over the beginning point.
2. When you name a plane with three points, choose no more than two collinear points.
3. An arrow indicates the direction of a path that extends without end.
4. A plane is represented by a parallelogram. However, the plane actually has no edges. It is flat and extends forever in all directions.

**Exercises**

Identify each figure as a point, segment, ray, line, or plane, and name each.

1. Plane; Answers may vary. Sample: plane \( LMN \)
2. Line; \( CD \)
3. Line; answers may vary. Sample: \( FG \)
4. Segment; \( PO \)
5. Ray; \( ST \)
A postulate is a statement that is accepted as true.

Postulate 1–4 states that through any three noncollinear points, there is only one plane. Noncollinear points are points that do not all lie on the same line.

In the figure at the right, points $D$, $E$, and $F$ are noncollinear. These points all lie in one plane.

Three noncollinear points lie in only one plane. Three points that are collinear can be contained by more than one plane. In the figure at the right, points $P$, $Q$, and $R$ are collinear, and lie in both plane $O$ and plane $N$.

**Exercises**

Identify the plane containing the given points as front, back, left side, right side, top, or bottom.

7. $F$, $G$, and $X$ **back**
8. $F$, $G$, and $H$ **top**
9. $H$, $I$, and $Z$ **front**
10. $F$, $W$, and $X$ **back**
11. $I$, $W$, and $Z$ **left side**
12. $Z$, $X$, and $Y$ **bottom**
13. $H$, $G$, and $X$ **right side**
14. $W$, $Y$, and $Z$ **bottom**

Use the figure at the right to determine how many planes contain the given group of points. Note that $\overrightarrow{GF}$ pierces the plane at $R$, $\overrightarrow{GF}$ is not coplanar with $X$, and $\overrightarrow{GF}$ does not intersect $\overrightarrow{CE}$.

15. $C$, $D$, and $E$ **infinite number of planes**
16. $D$, $E$, and $F$ **1 plane**
17. $C$, $G$, $E$, and $F$ **0 planes**
18. $C$ and $F$ **infinite number of planes**
The Segment Addition Postulate allows you to use known segment lengths to find unknown segment lengths. If three points, \( A \), \( B \), and \( C \), are on the same line, and point \( B \) is between points \( A \) and \( C \), then the distance \( AC \) is the sum of the distances \( AB \) and \( BC \).

\[
AC = AB + BC
\]

**Problem**

If \( QS = 7 \) and \( QR = 3 \), what is \( RS \)?

\[
QS = QR + RS \quad \text{Segment Addition Postulate}
\]
\[
QS - QR = RS \quad \text{Subtract} \ QR \ \text{from each side.}
\]
\[
7 - 3 = RS \quad \text{Substitute.}
\]
\[
4 = RS \quad \text{Simplify.}
\]

**Exercises**

For Exercises 1–5, use the figure at the right.

1. If \( PN = 29 \text{ cm} \) and \( MN = 13 \text{ cm} \), then \( PM = \) \( 16 \text{ cm} \).
2. If \( PN = 34 \text{ cm} \) and \( MN = 19 \text{ cm} \), then \( PM = \) \( 15 \text{ cm} \).
3. If \( PM = 19 \) and \( MN = 23 \), then \( PN = \) \( 42 \).
4. If \( MN = 82 \) and \( PN = 105 \), then \( PM = \) \( 23 \).
5. If \( PM = 100 \) and \( MN = 100 \), then \( PN = \) \( 200 \).

For Exercises 6–8, use the figure at the right.

6. If \( UW = 13 \text{ cm} \) and \( UX = 46 \text{ cm} \), then \( WX = \) \( 33 \text{ cm} \).
7. \( UW = 2 \) and \( UX = y \). Write an expression for \( WX \). \( y - 2 \)
8. \( UW = m \) and \( WX = y + 14 \). Write an expression for \( UX \). \( m + y + 14 \)

On a number line, the coordinates of \( A, B, C, \) and \( D \) are \(-6, -2, 3, \) and \( 7 \), respectively. Find the lengths of the two segments. Then tell whether they are congruent.

9. \( \overline{AB} \) and \( \overline{CD} \) \( 4; \) \( 4; \) yes
10. \( \overline{AC} \) and \( \overline{BD} \) \( 9; \) \( 9; \) yes
11. \( \overline{BC} \) and \( \overline{AD} \) \( 5; \) \( 13; \) no
The midpoint of a line segment divides the segment into two segments that are equal in length. If you know the distance between the midpoint and an endpoint of a segment, you can find the length of the segment. If you know the length of a segment, you can find the distance between its endpoint and midpoint.

\[ X \text{ is the midpoint of } WY. \quad XW = XY, \text{ so } \overline{WX} \text{ and } \overline{XY} \text{ are congruent.} \]

**Problem**

\[ C \text{ is the midpoint of } BE. \text{ If } BC = t + 1, \text{ and } CE = 15 - t, \text{ what is } BE? \]

\[
\begin{align*}
BC &= CE \\
t + 1 &= 15 - t \\
t + t + 1 &= 15 - t + t \\
2t + 1 &= 15 \\
2t + 1 - 1 &= 15 - 1 \\
2t &= 14 \\
t &= 7
\end{align*}
\]

\[
\begin{align*}
BC &= t + 1 \\
BC &= 7 + 1 \\
BC &= 8 \\
BE &= 2(BC) \\
BE &= 2(8) \\
BE &= 16
\end{align*}
\]

**Exercises**

12. \( W \) is the midpoint of \( \overline{UV} \). If \( UW = x + 23 \), and \( WV = 2x + 8 \), what is \( x? \) 15

13. \( W \) is the midpoint of \( \overline{UV} \). If \( UW = x + 23 \), and \( WV = 2x + 8 \), what is \( WU? \) 38

14. \( W \) is the midpoint of \( \overline{UV} \). If \( UW = x + 23 \), and \( WV = 2x + 8 \), what is \( UV? \) 76

15. \( Z \) is the midpoint of \( \overline{YA} \). If \( YZ = x + 12 \), and \( ZA = 6x - 13 \), what is \( YA? \) 34
1-4 Reteaching
Measuring Angles

The vertex of an angle is the common endpoint of the rays that form the angle. An angle may be named by its vertex. It may also be named by a number or by a point on each ray and the vertex (in the middle).

This is \( \angle Z, \angle XZY, \angle YZX, \) or \( \angle 1. \)

It is not \( \angle ZYX, \angle XYZ, \angle YXZ, \) or \( \angle ZXY. \)

Angles are measured in degrees, and the measure of an angle is used to classify it.

The measure of an acute angle is between 0 and 90.

The measure of a right angle is 90.

The measure of an obtuse angle is between 90 and 180.

The measure of a straight angle is 180.

Exercises

Use the figure at the right for Exercises 1 and 2.

1. What are three other names for \( \angle S? \) \( \angle RST, \angle TSR, \angle 3 \)

2. What type of angle is \( \angle S? \) right

3. Name the vertex of each angle.
   a. \( \angle LGH \) point \( G \)
   b. \( \angle MBX \) point \( B \)

Classify the following angles as acute, right, obtuse, or straight.

4. \( m\angle LGH = 14 \) acute

5. \( m\angle SRT = 114 \) obtuse

6. \( m\angle SLI = 90 \) right

7. \( m\angle 1 = 139 \) obtuse

8. \( m\angle L = 179 \) obtuse

9. \( m\angle P = 73 \) acute

Use the diagram below for Exercises 10–18.
Find the measure of each angle.

10. \( \angle ADB \) 25

11. \( \angle FDE \) 35

12. \( \angle BDC \) 55

13. \( \angle CDE \) 65

14. \( \angle ADC \) 80

15. \( \angle FDC \) 100

16. \( \angle BDE \) 120

17. \( \angle ADE \) 145

18. \( \angle BDF \) 155
The Angle Addition Postulate allows you to use a known angle measure to find an unknown angle measure. If point B is in the interior of \(\angle AXC\), the sum of \(m\angle AXB\) and \(m\angle BXC\) is equal to \(m\angle AXC\).

\[
m\angle AXB + m\angle BXC = m\angle AXC
\]

**Problem**

If \(m\angle LYN = 125\), what are \(m\angle LYM\) and \(m\angle MYN\)?

**Step 1** Solve for \(p\).

\[
m\angle LYN = m\angle LYM + m\angle MYN \quad \text{Angle Addition Postulate}
\]

\[
125 = (4p + 7) + (2p - 2) \quad \text{Substitute.}
\]

\[
125 = 6p + 5
\]

\[
120 = 6p \quad \text{Simplify.}
\]

\[
20 = p \quad \text{Divide each side by 6.}
\]

**Step 2** Use the value of \(p\) to find the measures of the angles.

\[
m\angle LYM = 4p + 7 \quad \text{Given}
\]

\[
m\angle LYM = 4(20) + 7 \quad \text{Substitute.}
\]

\[
m\angle LYM = 87 \quad \text{Simplify.}
\]

\[
m\angle MYN = 2p - 2 \quad \text{Given}
\]

\[
m\angle MYN = 2(20) - 2 \quad \text{Substitute.}
\]

\[
m\angle MYN = 38 \quad \text{Simplify.}
\]

**Exercises**

19. \(X\) is in the interior of \(\angle LIN\). \(m\angle LIN = 100\), \(m\angle LIX = 14t\), and \(m\angle XIN = t + 10\).
   a. What is the value of \(t\)? \(\text{6}\) \hfill b. What are \(m\angle LIX\) and \(m\angle XIN\)? \(\text{84, 16}\)

20. \(Z\) is in the interior of \(\angle GHI\). \(m\angle GHI = 170\), \(m\angle GHZ = 3s - 5\), and \(m\angle ZHI = 2s + 25\).
   a. What is the value of \(s\)? \(\text{30}\) \hfill b. What are \(m\angle GHZ\) and \(m\angle ZHI\)? \(\text{85, 85}\)
1-5 Reteaching
Exploring Angle Pairs

Adjacent Angles and Vertical Angles

*Adjacent* means “next to.” Angles are adjacent if they lie next to each other. In other words, the angles have the same vertex and they share a side without overlapping.

![Adjacent Angles](image1)

*Vertical* means “related to the vertex.” So, angles are vertical if they share a vertex, but not just any vertex. They share a vertex formed by the intersection of two straight lines. Vertical angles are always congruent.

![Vertical Angles](image2)

Exercises

1. Use the diagram at the right.
   a. Name an angle that is adjacent to $\angle ABE$.
      $\angle EBF$ or $\angle EBC$
   b. Name an angle that overlaps $\angle ABE$.
      Answers may vary.
      Sample: $\angle DBF$ or $\angle DBC$

2. Use the diagram at the right.
   a. Mark $\angle DOE$ and its vertical angle as congruent angles.
      $\angle AOB$
   b. Mark $\angle AOE$ and its vertical angle as congruent angles.
      $\angle DOB$
Supplementary Angles and Complementary Angles

Two angles that form a line are supplementary angles. Another term for these angles is a linear pair. However, any two angles with measures that sum to 180 are also considered supplementary angles. In both figures below, \( m \angle 1 = 120 \) and \( m \angle 2 = 60 \), so \( \angle 1 \) and \( \angle 2 \) are supplementary.

Two angles that form a right angle are complementary angles. However, any two angles with measures that sum to 90 are also considered complementary angles. In both figures below, \( m \angle 1 = 60 \) and \( m \angle 2 = 30 \), so \( \angle 1 \) and \( \angle 2 \) are complementary.

Exercises

3. Copy the diagram at the right.
   a. Label \( \angle ABD \) as \( \angle 1 \). \( \angle ABD \) as \( \angle 1 \).
   b. Label an angle that is supplementary to \( \angle ABD \) as \( \angle 2 \). \( \angle DBC \) as \( \angle 2 \).
   c. Label as \( \angle 3 \) an angle that is adjacent and complementary to \( \angle ABD \). \( \angle DBE \) as \( \angle 3 \).
   d. Label as \( \angle 4 \) a second angle that is complementary to \( \angle ABD \). \( \angle FBC \) as \( \angle 4 \).
   e. Name an angle that is supplementary to \( \angle ABE \). \( \angle CBE \)
   f. Name an angle that is complementary to \( \angle EBF \). \( \angle CBF \)
Before you start the construction, THINK about the goal. Then SKETCH the segment or angle you are trying to construct. Next EXPLAIN the purpose of each step in the construction as you complete it.

**Problem**

Construct $\overline{AB}$ so that $\overline{AB}$ is congruent to $\overline{XY}$.

THINK: Can you describe what the problem is asking for in your own words? *You want to draw a line segment that has the same length as one you are given.*

SKETCH: Sketch a segment and label it as $\overline{AB}$. Why do you start with a sketch? *A sketch helps you to see what you need to construct.*

EXPLAIN: First, draw a ray. What is the purpose of drawing a ray? *The ray is a part of a line on which you can mark off the correct length of $\overline{AB}.*

Second, measure the length of $\overline{XY}$, using the compass. What is the purpose of measuring the length of $\overline{XY}$? *You need this measure to mark the same length on the ray.*

Finally, mark the length of $\overline{XY}$ on the ray, using the compass. Why do you mark the length of $\overline{XY}$ on your ray? *This completes your construction of $\overline{AB}$, which is congruent to $\overline{XY}.*

**Exercises**

Analyze the construction of a congruent angle and bisectors.

1. Analyze the construction of a perpendicular bisector. First draw $\overline{XY}$.
   a. THINK about what it means to construct a perpendicular bisector. What is your goal? *Answers may vary. Sample: to draw a segment that is perpendicular to $\overline{XY}$ and that divides it into two equal pieces*
   b. SKETCH a perpendicular bisector to $\overline{XY}$. *Check students’ work.*
   c. EXPLAIN the first two steps. What is the purpose of drawing an arc from each endpoint? *Answers may vary. Sample: to form two intersections to become two points of the perpendicular bisector*
   d. EXPLAIN the last step. What is the purpose of drawing the segment between the intersections of the arcs? *Answers may vary. Sample: to draw the perpendicular bisector*
2. Analyze the construction of a congruent angle. First draw $\angle X$.
   a. THINK about what it means to construct a congruent angle. What is your goal?
      
      *Answers may vary. Sample: to draw an angle of the same size as the given angle*
   
   b. SKETCH $\angle Y$ congruent to $\angle X$. Your goal is to construct an angle the same size as $\angle Y$. Check students’ work.

   c. EXPLAIN the first step (drawing a ray). What is the purpose of the first step?
      
      *Answers may vary. Sample: to make a ray to serve as one side of the new congruent angle*
   
   d. EXPLAIN the second step (drawing an arc). What is the purpose of the second step?
      
      *Answers may vary. Sample: to measure the opening of the given angle*

   e. EXPLAIN the third step (drawing an arc). What is the purpose of the third step?
      
      *Answers may vary. Sample: to mark the size of the given angle on the ray*

   f. EXPLAIN the fourth step (drawing an arc). What is the purpose of the fourth step?
      
      *Answers may vary. Sample: to find the point of intersection so you can draw the other side of the angle*

   g. EXPLAIN the fifth step (drawing a segment). What is the purpose of the fifth step?
      
      *Answers may vary. Sample: to draw the congruent angle*

3. Analyze the construction of an angle bisector. First draw $\angle W$.
   a. THINK about what it means to construct an angle bisector. What is your goal?
      
      *Answers may vary. Sample: to divide a given angle into two equal angles*

   b. SKETCH a ray that makes $\angle V$ congruent to $\frac{1}{2} \angle W$, where $\angle V$ shares a ray with $\angle W$ and has its other ray inside $\angle W$. Your goal is to construct a ray that bisects an angle. Check students’ work.

   c. EXPLAIN the first step (drawing an arc). What is the purpose of the first step?
      
      *Answers may vary. Sample: to locate two points of intersection to use to draw the other endpoint of the angle bisector*

   d. EXPLAIN the second step (drawing two arcs). What is the purpose of the second step?
      
      *Answers may vary. Sample: to locate the point of intersection that will be another point on the angle bisector*

   e. EXPLAIN the third step (drawing a ray). What is the purpose of the third step?
      
      *Answers may vary. Sample: to divide the given angle into two equal smaller angles*
Average the $x$-coordinates of the endpoints to find the $x$-coordinate of the midpoint. Average the $y$-coordinates of the endpoints to find the $y$-coordinate of the midpoint.

**Problem**

What is the midpoint of $\overline{AB}$ if the endpoints are $A(1, 7)$ and $B(5, 9)$?

Find the average of the $x$-coordinates.

$$\frac{1 + 5}{2} = 3$$

Repeat to find the $y$-coordinate of the midpoint.

$$\frac{7 + 9}{2} = 8$$

So, the midpoint of $\overline{AB}$ is $(3, 8)$.

Remember the Midpoint Formula: $\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

The formula gives a point whose coordinates are the average of the $x$-coordinates and the $y$-coordinates.

So, the midpoint is halfway between the two points, and has coordinates that are the average of the coordinates of the two points.

To find an unknown endpoint, subtract the coordinates of the known endpoint from the coordinates of the midpoint. Add that number to the coordinates of the midpoint.

**Problem**

The midpoint of $\overline{XY}$ is $M(7, 6)$. One endpoint is $X(3, 5)$. What are the coordinates of the other endpoint $Y$?
Exercises

Find the coordinates of the midpoint of \(\overline{AB}\) by finding the averages of the coordinates.

1. \(A(4, 3)\) \(M(6, 5.5)\) \(B(8, 8)\)
2. \(A(7, 2)\) \(M(4, 3.5)\) \(B(1, 5)\)
3. \(A(-5, 6)\) \(M(-2, 1.5)\) \(B(1, -3)\)
4. \(A(-7, -1)\) \(M(-6, -5)\) \(B(-5, -9)\)

\(M\) is the midpoint of \(\overline{XY}\). Find the coordinates of \(Y\).

5. \(X(3, 4)\) and \(M(6, 10)\) \((9, 16)\)
6. \(X(-5, 1)\) and \(M(3, -5)\) \((-11, -11)\)

To help find the distance between two points, make a sketch on graph paper.

**Problem**

What is the distance between \(A(2, 6)\) and \(B(6, 9)\)?

**Step 1:** Sketch the points on graph paper.

**Step 2:** Draw a right triangle along the gridlines. \(\sqrt{3^2 + 4^2} = 5\).

**Step 3:** Find the length of each leg.

**Step 4:** Find the distance between the points.

Exercises

Find the distance between points \(A\) and \(B\). If necessary, round to the nearest tenth.

7. \(A(1, 4)\) and \(B(6, 16)\) \(13\)
8. \(A(-3, 2)\) and \(B(1, 6)\) \(5.7\)
9. \(A(-1, -8)\) and \(B(1, -3)\) \(5.4\)
10. \(A(-5, -5)\) and \(B(7, 11)\) \(20\)

Find the midpoint between each pair of points. Then, find the distance between each pair of points. If necessary, round to the nearest tenth.

11. \(C(3, 8)\) and \(D(0, 3)\) \((1.5, 5.5); 5.8\)
12. \(H(-2, 4)\) and \(I(4, -2)\) \((1, 1); 8.5\)
13. \(K(1, -5)\) and \(L(-3, -9)\) \((-1, -7); 5.7\)
14. \(M(7, 0)\) and \(N(-3, 4)\) \((2, 2); 10.8\)
15. \(O(-5, -1)\) and \(P(-2, 3)\) \((-3.5, 1); 5\)
16. \(R(0, -6)\) and \(S(-8, 0)\) \((-4, -3); 10\)
The perimeter of a rectangle is the sum of the lengths of its sides. So, the perimeter is the distance around its outside. The formula for the perimeter of a rectangle is $P = 2(b + h)$.

The area of a rectangle is the number of square units contained within the rectangle. The formula for the area of a rectangle is $A = bh$.

**Exercises**

1. Fill in the missing information for each rectangle in the table below.

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Perimeter, $P = 2(b + h)$</th>
<th>Area, $A = bh$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ft × 9 ft</td>
<td>$(1 \text{ ft} + 9 \text{ ft}) = 20 \text{ ft}$</td>
<td>$1 \text{ ft} \times 9 \text{ ft} = 9 \text{ ft}^2$</td>
</tr>
<tr>
<td>2 ft × 8 ft</td>
<td>$(2 \text{ ft} + 8 \text{ ft}) = 20 \text{ ft}$</td>
<td>$2 \text{ ft} \times 8 \text{ ft} = 16 \text{ ft}^2$</td>
</tr>
<tr>
<td>3 ft × 7 ft</td>
<td>$(3 \text{ ft} + 7 \text{ ft}) = 20 \text{ ft}$</td>
<td>$3 \text{ ft} \times 7 \text{ ft} = 21 \text{ ft}^2$</td>
</tr>
<tr>
<td>4 ft × 6 ft</td>
<td>$(4 \text{ ft} + 6 \text{ ft}) = 20 \text{ ft}$</td>
<td>$4 \text{ ft} \times 6 \text{ ft} = 24 \text{ ft}^2$</td>
</tr>
</tbody>
</table>

2. How does the perimeter vary as you move down the table? How does the area vary as you move down the table? **The perimeter stays the same. The area increases.**

3. What pattern in the dimensions of the rectangles explains your answer to Exercise 2? **Answers may vary. Sample: The sum $b + h$ is always the same, while the product $bh$ increases.**

4. Fill in the missing information for each rectangle in the table below.

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Perimeter, $P = 2(b + h)$</th>
<th>Area, $A = bh$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ft × 24 ft</td>
<td>$(1 \text{ ft} + 24 \text{ ft}) = 50 \text{ ft}$</td>
<td>$1 \text{ ft} \times 24 \text{ ft} = 24 \text{ ft}^2$</td>
</tr>
<tr>
<td>2 ft × 12 ft</td>
<td>$(2 \text{ ft} + 12 \text{ ft}) = 28 \text{ ft}$</td>
<td>$2 \text{ ft} \times 12 \text{ ft} = 24 \text{ ft}^2$</td>
</tr>
<tr>
<td>3 ft × 8 ft</td>
<td>$(3 \text{ ft} + 8 \text{ ft}) = 22 \text{ ft}$</td>
<td>$3 \text{ ft} \times 8 \text{ ft} = 24 \text{ ft}^2$</td>
</tr>
<tr>
<td>4 ft × 6 ft</td>
<td>$(4 \text{ ft} + 6 \text{ ft}) = 20 \text{ ft}$</td>
<td>$4 \text{ ft} \times 6 \text{ ft} = 24 \text{ ft}^2$</td>
</tr>
</tbody>
</table>

5. How does the perimeter vary as you move down the table? How does the area vary as you move down the table? **The perimeter decreases. The area stays the same.**

6. What pattern in the dimensions of the rectangles explains your answer to Exercise 5? **Answers may vary. Sample: The product $bh$ is always the same, while the sum $b + h$ decreases.**
Perimeter, Circumference, and Area

A square is a rectangle that has four sides of the same length. Because the perimeter is \( s + s + s + s \), the formula for the perimeter of a square is \( P = 4s \). The formula for the area of a square is \( A = s^2 \).

The circumference of a circle is the distance around the circle. The formula for the circumference of a circle is \( C = \pi d \) or \( C = 2\pi r \). The area of a circle is the number of square units contained within the circle. The formula for the area of a circle is \( A = \pi r^2 \).

Exercises

7. Fill in the missing information for each square in the table below.

<table>
<thead>
<tr>
<th>Side</th>
<th>Perimeter, ( P = 4s )</th>
<th>Area, ( A = s^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 cm</td>
<td>( 4 \times 3 \ cm = 12 \ cm )</td>
<td>( (3 \ cm)^2 = 9 \ cm^2 )</td>
</tr>
<tr>
<td>4 cm</td>
<td>( 4 \times 4 \ cm = 16 \ cm )</td>
<td>( (4 \ cm)^2 = 16 \ cm^2 )</td>
</tr>
<tr>
<td>5 cm</td>
<td>( 4 \times 5 \ cm = 20 \ cm )</td>
<td>( (5 \ cm)^2 = 25 \ cm^2 )</td>
</tr>
<tr>
<td>10 cm</td>
<td>( 4 \times 10 \ cm = 40 \ cm )</td>
<td>( (10 \ cm)^2 = 100 \ cm^2 )</td>
</tr>
</tbody>
</table>

8. Fill in the missing information for each circle in the table below.

<table>
<thead>
<tr>
<th>Radius</th>
<th>Diameter, ( D = 2r )</th>
<th>Circumference, ( C = 2\pi r )</th>
<th>Area, ( A = \pi r^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 in.</td>
<td>( 2 \times 2 = 4 \ in. )</td>
<td>( 2\pi \times 2 = 4\pi \ in. )</td>
<td>( \pi \times 2 \times 2 = 4\pi \ in. )</td>
</tr>
<tr>
<td>3 in.</td>
<td>( 2 \times 3 = 6 \ in. )</td>
<td>( 2\pi \times 3 = 6\pi \ in. )</td>
<td>( \pi \times 3 \times 3 = 9\pi \ in. )</td>
</tr>
<tr>
<td>5 in.</td>
<td>( 2 \times 5 = 10 \ in. )</td>
<td>( 2\pi \times 5 = 10\pi \ in. )</td>
<td>( \pi \times 5 \times 5 = 25\pi \ in. )</td>
</tr>
<tr>
<td>8 in.</td>
<td>( 2 \times 8 = 16 \ in. )</td>
<td>( 2\pi \times 8 = 16\pi \ in. )</td>
<td>( \pi \times 8 \times 8 = 64\pi \ in. )</td>
</tr>
<tr>
<td>10 in.</td>
<td>( 2 \times 10 = 20 \ in. )</td>
<td>( 2\pi \times 10 = 20\pi \ in. )</td>
<td>( \pi \times 10 \times 10 = 100\pi \ in. )</td>
</tr>
</tbody>
</table>

9. A rectangle has a length of 5 cm and an area of 20 cm\(^2\). What is its width? \( 4 \) cm

10. What is the perimeter of a square whose area is 81 ft\(^2\)? \( 36 \) ft

11. Can you find the perimeter of a rectangle if you only know its area? What about a square? Explain.

Answers may vary. Sample: It is impossible to determine the area of a rectangle if only the perimeter is known, because rectangles with the same area may have different perimeters. However, because all squares with the same area have the same perimeter, it is possible to determine the area if only the perimeter is known.

12. Can you find the area of a circle if you only know its circumference? Explain.

Answers may vary. Sample: Yes; all circles with the same circumference have the same area.
Inductive reasoning is a type of reasoning in which you look at a pattern and then make some type of prediction based on the pattern. These predictions are also called conjectures. A conjecture is a statement about what you think will happen based on the pattern you observed.

### Problem

Which conjectures below are reasonable? Which are not?

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Conjecture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern 1: Every day for the two weeks that Alba visited Cairo, the weather was hot and dry.</td>
<td>Alba thought to herself, “The weather is always hot and dry in this city.”</td>
</tr>
<tr>
<td>Pattern 2: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, . . .</td>
<td>Each number increases by 5. The next two numbers are most likely 55 and 60.</td>
</tr>
<tr>
<td>Pattern 3: Dani and Liz both examined this pattern of letters: A, BB, CCC, . . .</td>
<td>Dani was sure that the next two terms of the pattern would be DDDD and EEEE. Liz wasn’t so sure. She thought the pattern might repeat and the next two elements would be A and BB.</td>
</tr>
</tbody>
</table>

**Pattern 1:** The conjecture for Pattern 1 is probably not correct. In most cities, the weather will not be hot and dry all the time.

**Pattern 2:** You are given enough numbers in the pattern to assume that the numbers continue to increase by five. The conjecture is probably correct.

**Pattern 3:** Only three terms of the pattern are shown. This makes it difficult to determine what rule the pattern follows. Either Dani or Liz could be correct, or they could both be incorrect.

Remember that conjectures are never the final goal in a complete reasoning process. They are simply the first step to figuring out a problem.

### Exercises

Make a conjecture about the rule these patterns follow.

1. 3, 6, 9, 12, 15, . . .
   - The numbers increase by 3.
2. 9, 3, 1, , 1, . . .
   - The numbers are divided by 3.
   - One letter of the alphabet is skipped between each term.
4. 0, 5, −2, 3, −4, 1, −6, . . .
   - Add 5, then subtract 7.
5. 4, 8, 16, 32, 64, 128, . . .
   - The numbers are multiplied by 2.
6. 0.1, 0.01, 0.001, 0.0001, . . .
   - The numbers are divided by 10.
2-1 Reteaching (continued)
Patterns and Inductive Reasoning

Draw the next figure in each sequence.

7. ____________________________
8. ____________________________

One way to show that a conjecture is not true is to find a counterexample. A counterexample is an instance in which the conjectured pattern does not work. Only one counterexample is needed to prove a conjecture false. For example, one rainy or cool day in Cairo would prove to Alba that it is not always hot and dry there.

Exercises

Find one counterexample to show that each conjecture is false.

9. All vehicles on the highway have exactly four wheels.
   Sample: Motorcycles have two wheels.

10. All states in the United States share a border with another state.
    Sample: Hawaii

11. All plurals end with the letter s.
    Sample: geese

12. The difference between two integers is always positive.
    Sample: \(-3 - 2 = -5\)

13. All pentagons have exactly five congruent sides.
    Sample: an irregular pentagon

14. All numbers that are divisible by 3 are also divisible by 6.
    Sample: 15

15. All whole numbers are greater than their opposites.
    0

16. All prime numbers are odd integers.
    2
2-2  Reteaching
Conditional Statements

A conditional is a statement with an “if” clause and a “then” clause. Here are some examples of conditional statements:

If the flag is extended all the way outward, then it is very windy.
If a cup is in pieces, then it is broken.

A conditional is divided into two parts—the hypothesis and the conclusion.

If Alex decides to eat dessert, then Alex will eat apple pie.

<table>
<thead>
<tr>
<th>Hypothesis: the conditions necessary to reach the conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conclusion: the result if the conditions from the hypothesis are present</td>
</tr>
</tbody>
</table>

In the above conditional, the hypothesis is: Alex decides to eat dessert. The conclusion is: Alex will eat apple pie.

Conditionals can be changed to form related statements. Besides the original conditional, there are also the converse, inverse, and contrapositive of a conditional.

**Problem**

Write the converse, inverse, and contrapositive of the following statement:

If the weather is rainy, then the sidewalks will be wet.

To write the converse of a conditional, write the conclusion as the hypothesis and the hypothesis as the conclusion.

Conditional: If the weather is rainy, then the sidewalks are wet.

Converse: If the sidewalks are wet, then the weather is rainy.

To write the inverse of a conditional, negate the original hypothesis and conclusion.

Conditional: If the weather is rainy, then the sidewalks are wet.

Inverse: If the weather is not rainy, then the sidewalks are not wet.
To write the contrapositive of a conditional, negate the original hypothesis and conclusion and also switch the hypothesis and conclusion.

**Conditional:** If the weather is rainy, then the sidewalks are wet.

**Contrapositive:** If the sidewalks are not wet, then the weather is not rainy.

The conditional and its contrapositive are always both true or both false, so they have the same truth value. The converse and the inverse also have the same truth value. However, the conditional and contrapositive may have a different truth value than the converse and the inverse.

**Exercises**

Identify the hypothesis and conclusion of each conditional.

1. If a number is a multiple of 2, then the number is even.
   - **Hypothesis:** A number is a multiple of 2. **Conclusion:** The number is even.

2. If something is thrown up into the air, then it must come back down.
   - **Hypothesis:** Something is thrown up into the air. **Conclusion:** It must come back down.

3. Two angles are supplementary if the angles form a linear pair.
   - **Hypothesis:** Two angles form a linear pair. **Conclusion:** The angles are supplementary.

4. If the shoe fits, then you should wear it.
   - **Hypothesis:** The shoe fits. **Conclusion:** You should wear it.

State whether each conditional is true or false. Write the converse for the conditional and state whether the converse is true or false.

5. If the recipe uses 3 teaspoons of sugar, then it uses 1 tablespoon of sugar.
   - True; if the recipe uses 1 tablespoon of sugar, then it uses 3 teaspoons of sugar; true.

6. If the milk has passed its expiration date, then the milk should not be consumed.
   - True; if the milk should not be consumed, then the milk has passed its expiration date; false.

State whether each conditional is true or false. Write the inverse for the conditional and state whether the inverse is true or false.

7. If the animal is a fish, then it lives in water.
   - True; if the animal is not a fish, then the animal does not live in water; false.

8. If your car tires are not properly inflated, then you will get lower gas mileage.
   - True; if your car tires are inflated properly, then you will not get lower gas mileage; true.

State whether each conditional is true or false. Write the contrapositive for the conditional and state whether the contrapositive is true or false.

9. If you ride on a roller coaster, then you will experience sudden drops.
   - True; if you do not experience sudden drops, then you are not riding a roller coaster; true.

10. If you only have $15, then you can buy a meal that costs $15.65.
    - False; if you cannot buy a meal that costs $15.65, then you do not only have $15; false.
If a conditional statement and its converse are both true, we say the original conditional is reversible. It works both ways because both $p \rightarrow q$ and $q \rightarrow p$ are true.

If a conditional is reversible, you can write it as a **biconditional**. A biconditional uses the words “if and only if.” A biconditional can be written as $p \leftrightarrow q$.

**Conditional:** If a triangle has three congruent sides, then the triangle is equilateral.

**Converse:** If a triangle is equilateral, then the triangle has three congruent sides.

The conditional is true. The converse is also true. Since the conditional and its converse are both true, the original statement is “reversible” and the biconditional will be true.

**Biconditional:** A triangle has three congruent sides if and only if it is an equilateral triangle.

### Exercises

Test each statement below to see if it is reversible. If it is reversible, write it as a true biconditional. If not, write **not reversible**.

1. If a whole number is a multiple of 2, then the whole number is even.
   - Reversible; a whole number is a multiple of 2 if and only if the whole number is even.
2. Rabbits are animals that eat carrots.
   - not reversible
3. Two lines that intersect to form four $90^{\circ}$ angles are perpendicular.
   - Reversible; two lines intersect to form four $90^{\circ}$ angles if and only if the two lines are perpendicular.
4. Mammals are warm-blooded animals.
   - not reversible

Write the two conditionals that form each biconditional.

5. A parallelogram is a rectangle if and only if the diagonals are congruent.
   - If a parallelogram is a rectangle, then the diagonals are congruent. If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.
6. An animal is a giraffe if and only if the animal’s scientific name is *Giraffa camelopardalis*.
   - If an animal is a giraffe, then the animal’s scientific name is *Giraffa camelopardalis*. If the animal’s scientific name is *Giraffa camelopardalis*, then the animal is a giraffe.
Biconditionals and Definitions

A good definition:
- is reversible; the statement works both ways
- can be written as a true biconditional
- avoids using vague, imprecise, or difficult words

**Problem**

Is the following a good definition for a square?

**Definition:** A square is a rectangle with four congruent sides.

Is the definition reversible? Yes.

A rectangle with four congruent sides is a square.

Can the definition be rewritten as a biconditional? Yes.

A figure is a square if and only if it is a rectangle with four congruent sides.

Is the definition clear and understandable? Yes.

The definition is a good definition for a square.

**Exercises**

State whether each statement is a good definition. Explain your answer.

7. A parallelogram is a quadrilateral with two pairs of parallel sides.
   *The definition is good. The statement is reversible. It can be written as a true biconditional and is clear and understandable.*

8. A triangle is a three-sided figure whose angle measures sum to 180°.
   *The definition is good. The statement is reversible. It can be written as a true biconditional and is clear and understandable.*

9. A juice drink is a beverage that contains less than 100% juice.
   *The definition is not good. The statement is not reversible because a beverage that contains 0% juice, such as water, should not be called a juice drink.*

10. In basketball, the top scorer in a game is the player who scores the most points in the game.
    *The definition is good. The statement is reversible and can be written as a true biconditional and is clear and understandable.*

11. A tree is a large, green, leafy plant.
    *The definition is not good because the term “large” is vague. There are large, green, leafy plants that are not trees.*
The Law of Detachment states that if a conditional statement is true, then any time the conditions for the hypothesis exist, the conclusion is true. If the conditional statement is not true, or the conditions of the hypothesis do not exist, then you cannot make a valid conclusion.

**Problem**

What can you conclude from the following series of statements?

If an animal has feathers and can fly, then it is a bird.
A crow has feathers and can fly.
   - Is the conditional statement true? Yes.
   - Do the conditions of the hypothesis exist? Yes.
   - Therefore, you can conclude that a crow is a bird.

If an animal has feathers and can fly, then it is a bird.
A bat can fly.
   - Is the conditional statement true? Yes.
   - Do the conditions of the hypothesis exist? No; a bat does not have feathers.
   - Therefore, no conclusions can be made with the given information.

**Exercises**

Use the Law of Detachment to make a valid conclusion based on each conditional. Assume the conditional statement is true.

1. If it is Monday, then Jim has tae kwon do practice.
   - The date is Monday, August 25.  **Conclusion: Jim has tae kwon do practice.**

2. If the animal is a whale, then the animal lives in the ocean.
   - Daphne sees a beluga whale.  **Conclusion: The beluga whale lives in the ocean.**

3. If you live in the city of Miami, then you live in the state of Florida.
   - Jani lives in Florida.  **Conclusion: No conclusion is possible.**

4. If a triangle has an angle with a measure greater than 90, then the triangle is obtuse.
   - In \( \triangle GHI \), \( m\angle HGI = 110 \).  **Conclusion: \( \triangle GHI \) is obtuse.**

5. A parallelogram is a rectangle if its diagonals are congruent.
   - Lincoln draws a parallelogram on a piece of paper.  **Conclusion: No conclusion is possible.**
2-4 Reteaching (continued)
Deductive Reasoning

You can use the Law of Syllogism to string together two or more conditionals and draw a conclusion based on the conditionals. Notice that the conclusion of each conditional becomes the hypothesis of the next conditional.

\[
\begin{align*}
& p \rightarrow q \\
& q \rightarrow r \\
\therefore & p \rightarrow s
\end{align*}
\]

**Problem**

What can you conclude from the following two conditionals?

If a polygon is a hexagon, then the sum of its angle measures is 720.

If a polygon’s angle measures sum to 720, then the polygon has six sides.

The conclusion of the first conditional matches the hypothesis of the second conditional: the sum of the angle measures of a polygon is 720.

You can conclude that if a polygon is a hexagon, then it has six sides.

**Exercises**

If possible, use the Law of Syllogism to make a conclusion. If it is not possible to make a conclusion, tell why.

6. If you are climbing Pikes Peak, then you are in Colorado. If you are in Colorado, then you are in the United States.  
\textbf{Conclusion: If you are climbing Pikes Peak, then you are in the United States.}

7. If the leaves are falling from the trees, then it is fall.

If it is September 30, then it is fall.

\textbf{No valid conclusion is possible. The two statements stand alone.}

8. If it is spring, then the leaves are coming back on the trees.

If it is April 14, then it is spring.

\textbf{Conclusion: If it is April 14, then the leaves are coming back on the trees.}

9. If plogs plunder, then flegs fret.

If flegs fret, then gops groan.

\textbf{Conclusion: If plogs plunder, then gops groan.}

10. If you make a 95 on the next test, you will pass the course.

If you pass the course, you will not have to take summer school.

\textbf{Conclusion: If you make a 95 on the next test, then you will not have to take summer school.}

11. Use the Law of Detachment and the Law of Syllogism to make a valid conclusion from the following series of true statements. Explain your reasoning and which statements you used in your reasoning.

\begin{enumerate}
\item If an animal is an insect, then it has a head, thorax, and abdomen.
\item If an animal is a spider, then it has a head and abdomen.
\item If an animal has a head, thorax, and abdomen, then it has six legs.
\item Humberto saw an insect called a grasshopper.
\end{enumerate}

\textbf{Sample: Conclusion: The grasshopper has a head, a thorax, an abdomen, and six legs. I used the Law of Detachment, the Law of Syllogism, and statements I, III, and IV.}
2-5 Reteaching
Reasoning in Algebra and Geometry

When you solve equations you use the Properties of Equality.

<table>
<thead>
<tr>
<th>Property</th>
<th>Words</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition Property</td>
<td>You can add the same number to each side of an equation.</td>
<td>If ( x = 2 ), then ( x + 2 = 4 ).</td>
</tr>
<tr>
<td>Subtraction Property</td>
<td>You can subtract the same number from each side of an equation.</td>
<td>If ( y = 8 ), then ( y - 3 = 5 ).</td>
</tr>
<tr>
<td>Multiplication Property</td>
<td>You can multiply by the same number on each side of an equation.</td>
<td>If ( z = 2 ), then ( 5z = 10 ).</td>
</tr>
<tr>
<td>Division Property</td>
<td>You can divide each side of an equation by the same number.</td>
<td>If ( 6m = 12 ), then ( m = 2 ).</td>
</tr>
<tr>
<td>Substitution Property</td>
<td>You can exchange a part of an expression with an equivalent value.</td>
<td>If ( 3x + 5 = 3 ) and ( x = 2 ), then ( 3(2) + 5 = 3 ).</td>
</tr>
</tbody>
</table>

Exercises

Support each conclusion with a reason.

1. Given: \( 6x + 2 = 12 \)
   
   Conclusion: \( 6x = 10 \)
   **Subtraction Property**

2. Given: \( m \angle 1 + m \angle 2 = 90 \)
   
   Conclusion: \( m \angle 1 = 90 - m \angle 2 \)
   **Subtraction Property**

3. Given: \( x = m \angle C \)
   
   Conclusion: \( 2x = m \angle C + x \)
   **Addition Property**

4. Given: \( q - x = r \)
   
   Conclusion: \( 4(q - x) = 4r \)
   **Multiplication Property**

5. Given: \( m \angle Q - m \angle R = 90 \),
   
   \( m \angle Q = 4m \angle R \)
   
   Conclusion: \( 4m \angle R - m \angle R = 90 \)
   **Substitution Property**

6. Given: \( CD = AF - 2CD \)
   
   Conclusion: \( 3CD = AF \)
   **Addition Property**

7. Given: \( 5(y - x) = 20 \)
   
   Conclusion: \( 5y - 5x = 20 \)
   **Distributive Prop.**

8. Given: \( m \angle AOX = 2m \angle XOB \)
   
   \( 2m \angle XOB = 140 \)
   
   Conclusion: \( m \angle AOX = 140 \)
   **Substitution Prop.**

9. Order the steps below to complete the proof.

   **Given:** \( m \angle P + m \angle Q = 90 \), \( m \angle Q = 5m \angle P \)
   
   **Prove:** \( m \angle Q = 75 \)
   
   \( d, a, c, b \)
   
   a) \( 6m \angle P = 90 \) by the Distributive Property
   
   b) \( m \angle Q = 5 \cdot 15 = 75 \) by the Substitution and Multiplication Properties
   
   c) \( m \angle P = 15 \) by the Division Property
   
   d) \( m \angle P + 5m \angle P = 90 \) by the Substitution Property
Several other important properties are also needed to write proofs.

### Exercises

Match the property to the appropriate statement.

10. $\overline{RT} \equiv \overline{RT}$ \hspace{1cm} b \hspace{1cm} a) Reflexive Property of Equality

11. If $\angle YER \equiv \angle IOP$ \hspace{1cm} f \hspace{1cm} b) Reflexive Property of Congruence

   and $\angle IOP \equiv \angle WXYZ$

   then $\angle YER \equiv \angle WXYZ$

12. If $\overline{PQ} \equiv \overline{MN}$ \hspace{1cm} d \hspace{1cm} c) Symmetric Property of Equality

   then $\overline{MN} \equiv \overline{PQ}$

13. If $XT = YZ$ \hspace{1cm} e \hspace{1cm} d) Symmetric Property of Congruence

   and $YZ = UP$

   then $XT = UP$

14. $m\angle 1 = m\angle 1$ \hspace{1cm} a \hspace{1cm} e) Transitive Property of Equality

15. If $m\angle RQS = m\angle TEF$ \hspace{1cm} c \hspace{1cm} f) Transitive Property of Congruence

   then $m\angle TEF = m\angle RQS$

16. **Writing** Write six new mathematical statements that represent each of the properties given above.

   Check students' work.
A *theorem* is a conjecture or statement that you prove true using deductive reasoning. You prove each step using any of the following: given information, definitions, properties, postulates, and previously proven theorems.

The proof is a chain of logic. Each step is justified, and then the Laws of Detachment and Syllogism connect the steps to prove the theorem.

Vertical angles are angles on opposite sides of two intersecting lines. In the figure at the right, \( \angle 1 \) and \( \angle 3 \) are vertical angles. \( \angle 2 \) and \( \angle 4 \) are also vertical angles. The Vertical Angles Theorem states that vertical angles are always congruent. The symbol \( \equiv \) means *is congruent to*.

**Problem**

Given: \( m\angle BOF = m\angle FOD \)

Prove: \( 2m\angle BOF = m\angle AOE \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( m\angle BOF = m\angle FOD )</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( m\angle BOF + m\angle FOD = m\angle BOD )</td>
<td>2) Angle Addition Postulate</td>
</tr>
<tr>
<td>3) ( m\angle BOF + m\angle BOF = m\angle BOD )</td>
<td>3) Substitution Property</td>
</tr>
<tr>
<td>4) ( 2(m\angle BOF) = m\angle BOD )</td>
<td>4) Combine like terms.</td>
</tr>
<tr>
<td>5) ( \angle AOE \equiv \angle BOD )</td>
<td>5) Vertical Angles are ( \equiv ).</td>
</tr>
<tr>
<td>6) ( m\angle AOE = m\angle BOD )</td>
<td>6) Definition of Congruence</td>
</tr>
<tr>
<td>7) ( 2m\angle BOF = m\angle AOE )</td>
<td>7) Substitution Property</td>
</tr>
</tbody>
</table>

**Exercises**

Write a paragraph proof.

1. Given: \( \angle AOB \) and \( \angle XOZ \) are vertical angles.

   \[
   m\angle AOB = 80 \\
   m\angle XOZ = 6x + 5
   \]

   Prove: \( x = 12.5 \)

   Because it is given that \( \angle AOB \) and \( \angle XOZ \) are vertical angles, and all vertical angles are \( \equiv \), it follows that \( m\angle AOB = m\angle XOZ \). Because it is also given that \( m\angle AOB = 80 \) and \( m\angle XOZ = 6x + 5 \), the use of substitution yields the equation \( 80 = 6x + 5 \). Using the Subtraction Property of Equality, \( 75 = 6x \), and using the Division Property of Equality, it follows that \( x = 12.5 \).
Find the value of each variable.

2. \((6y - 16)^\circ\) \((4y + 6)^\circ\) \(11\)

3. \((3x + 50)^\circ\) \((7x - 10)^\circ\) \(15\)

4. \((82 - 4g)^\circ\) \((7g + 5)^\circ\) \(7\)

You can use numbers to help understand theorems that may seem confusing.

**Congruent Supplements Theorem:** If two angles are supplements of the same angle (or of congruent angles), then the two angles are congruent.

Think about it: Suppose \(m \angle 1 = 50\). Any angle supplementary to \(\angle 1\) must have a measure of 130. So, supplements of \(\angle 1\) must be congruent.

If \(\angle 2\) and \(\angle 3\) are both supplementary to \(\angle 1\), then \(\angle 2 \cong \angle 3\).

**Congruent Complements Theorem:** If two angles are complements of the same angle (or of congruent angles), then the two angles are congruent.

Think about it: Suppose \(m \angle 6 = 30\). Any complement of \(\angle 6\) has a measure of 60. So, all complements of \(\angle 6\) must be congruent.

If \(\angle 4\) and \(\angle 5\) are both complementary to \(\angle 6\), then \(\angle 4 \cong \angle 5\).

**Exercises**

Name a pair of congruent angles in each figure. Justify your answer.

5. Given: \(\angle 2\) is complementary to \(\angle 3\).

6. Given: \(\angle AYZ \cong \angle BYW\)

7. Reasoning Explain why the following statement is true. Use numbers in your explanation. “If \(\angle 1\) is supplementary to \(\angle 2\), \(\angle 2\) is supplementary to \(\angle 3\), \(\angle 3\) is supplementary to \(\angle 4\), and \(\angle 4\) is supplementary to \(\angle 5\), then \(\angle 1 \cong \angle 5\).”

Suppose \(m \angle 1 = 50\); therefore \(m \angle 2 = 130\), \(m \angle 3 = 50\), \(m \angle 4 = 130\), and \(m \angle 5 = 50\). Because \(m \angle 1 = m \angle 5\), \(\angle 1 \cong \angle 5\).
3-1 Reteaching
Lines and Angles

Not all lines and planes intersect.
- Planes that do not intersect are parallel planes.
- Lines that are in the same plane and do not intersect are parallel.
- The symbol \( \parallel \) shows that lines or planes are parallel: \( AB \parallel CD \) means “Line \( AB \) is parallel to line \( CD \).”
- Lines that are not in the same plane and do not intersect are skew.

Parallel planes: plane \( ABDC \parallel plane \ EFHG \)
- plane \( BFHD \parallel plane \ AEGC \)
- plane \( CDHG \parallel plane \ ABFE \)

Examples of parallel lines:
- \( \overrightarrow{CD} \parallel \overrightarrow{AB} \parallel \overrightarrow{EF} \parallel \overrightarrow{GH} \)

Examples of skew lines:
- \( \overrightarrow{CD} \) is skew to \( \overrightarrow{BF} \), \( \overrightarrow{AE} \), \( \overrightarrow{EG} \), and \( \overrightarrow{FH} \).

Exercises

In Exercises 1–3, use the figure at the right. Answers may vary. Sample:
1. Shade one set of parallel planes.
2. Trace one set of parallel lines with a solid line.
3. Trace one set of skew lines with a dashed line.

In Exercises 4–7, use the diagram to name each of the following.
4. a line that is parallel to \( \overrightarrow{RS} \)
   Answers may vary. Sample: \( \overrightarrow{NO} \)
5. a line that is skew to \( \overrightarrow{QU} \)
   Answers may vary. Sample: \( \overrightarrow{RS} \)
6. a plane that is parallel to \( NRTP \)
   plane \( OSUQ \)
7. three lines that are parallel to \( \overrightarrow{OQ} \)
   \( \overrightarrow{NP} \), \( \overrightarrow{RT} \), and \( \overrightarrow{SU} \)

In Exercises 8–11, describe the statement as true or false.
If false, explain.
8. plane \( HIJK \parallel plane \ IEGK \)
   False; these planes intersect.
9. \( \overrightarrow{DH} \parallel \overrightarrow{GK} \)
   True
10. \( \overrightarrow{HJ} \) and \( \overrightarrow{HD} \) are skew lines.
    False; the lines intersect.
11. \( \overrightarrow{FG} \parallel \overrightarrow{KI} \)
    False; the lines are in different planes, and since they do not intersect they are skew.
The diagram shows lines \( a \) and \( b \) intersected by line \( x \).

Line \( x \) is a transversal. A transversal is a line that intersects two or more lines found in the same plane.

The angles formed are either interior angles or exterior angles.

### Interior Angles

\( \angle 3, \angle 4, \angle 5, \text{ and } \angle 6 \) in diagram above outside angles on opposite sides of the transversal, not a linear pair

### Exterior Angles

\( \angle 1, \angle 2, \angle 7, \text{ and } \angle 8 \) in diagram above outside angles on opposite sides of the transversal, not a linear pair

Four types of special angle pairs are also formed.

<table>
<thead>
<tr>
<th>Angle Pair</th>
<th>Definition</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>alternate interior</td>
<td>inside angles on opposite sides of the transversal, not a linear pair</td>
<td>( \angle 3 ) and ( \angle 6 ) ( \angle 4 ) and ( \angle 5 )</td>
</tr>
<tr>
<td>alternate exterior</td>
<td>outside angles on opposite sides of the transversal, not a linear pair</td>
<td>( \angle 1 ) and ( \angle 8 ) ( \angle 2 ) and ( \angle 7 )</td>
</tr>
<tr>
<td>same-side interior</td>
<td>inside angles on the same side of the transversal</td>
<td>( \angle 3 ) and ( \angle 5 ) ( \angle 4 ) and ( \angle 6 )</td>
</tr>
<tr>
<td>corresponding</td>
<td>in matching positions above or below the transversal, but on the same side of the transversal</td>
<td>( \angle 1 ) and ( \angle 5 ) ( \angle 3 ) and ( \angle 7 ) ( \angle 2 ) and ( \angle 6 ) ( \angle 4 ) and ( \angle 8 )</td>
</tr>
</tbody>
</table>

### Exercises

Use the diagram at the right to answer Exercises 12–15.

12. Name all pairs of corresponding angles.
   \( \angle 1 \) and \( \angle 5 \); \( \angle 2 \) and \( \angle 6 \); \( \angle 3 \) and \( \angle 8 \); \( \angle 4 \) and \( \angle 7 \)

13. Name all pairs of alternate interior angles.
   \( \angle 4 \) and \( \angle 6 \); \( \angle 3 \) and \( \angle 5 \)

14. Name all pairs of same-side interior angles.
   \( \angle 4 \) and \( \angle 5 \); \( \angle 3 \) and \( \angle 6 \)

15. Name all pairs of alternate exterior angles.
   \( \angle 1 \) and \( \angle 8 \); \( \angle 2 \) and \( \angle 7 \)

Use the diagram at the right to answer Exercises 16 and 17. Decide whether the angles are alternate interior angles, same-side interior angles, corresponding, or alternate exterior angles.

16. \( \angle 1 \) and \( \angle 5 \) alternate exterior angles

17. \( \angle 4 \) and \( \angle 6 \) same-side interior angles
When a transversal intersects parallel lines, special congruent and supplementary angle pairs are formed.

**Congruent angles formed by a transversal intersecting parallel lines:**
- corresponding angles (Postulate 3-1)
  \( \angle 1 \cong \angle 5 \quad \angle 2 \cong \angle 6 \)
  \( \angle 4 \cong \angle 7 \quad \angle 3 \cong \angle 8 \)
- alternate interior angles (Theorem 3-1)
  \( \angle 4 \cong \angle 6 \quad \angle 3 \cong \angle 5 \)
- alternate exterior angles (Theorem 3-3)
  \( \angle 1 \cong \angle 8 \quad \angle 2 \cong \angle 7 \)

**Supplementary angles formed by a transversal intersecting parallel lines:**
- same-side interior angles (Theorem 3-2)
  \( m\angle 4 + m\angle 5 = 180 \quad m\angle 3 + m\angle 6 = 180 \)

**Identify all the numbered angles congruent to the given angle. Explain.**

1. \( \angle 6; \text{ vert. } \angle \text{ are } \cong; \angle 2; \) corr esp. \( \triangle \text{ are } \cong; \angle 4; \) alt. ext. \( \triangle \text{ are } \cong. \)
2. \( \angle 2; \text{ vert. } \angle \text{ are } \cong; \angle 5; \) alt int. \( \angle \text{ are } \cong; \angle 7; \) corr esp. \( \angle \text{ are } \cong. \)

**4. Supply the missing reasons in the two-column proof.**

**Given:** \( g \parallel h, i \parallel j \)

**Prove:** \( \angle 1 \text{ is supplementary to } \angle 16. \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( \angle 1 \cong \angle 3 )</td>
<td>1) ? Vertical angles are congruent.</td>
</tr>
<tr>
<td>2) ( g \parallel h; i \parallel j )</td>
<td>2) Given</td>
</tr>
<tr>
<td>3) ( \angle 3 \cong \angle 11 )</td>
<td>3) ? Corresponding angles are congruent.</td>
</tr>
<tr>
<td>4) ( \angle 11 \text{ and } \angle 16 \text{ are supplementary.} )</td>
<td>4) ? Same-side interior angles are supplementary.</td>
</tr>
<tr>
<td>5) ( \angle 1 \text{ and } \angle 16 \text{ are supplementary.} )</td>
<td>5) ? Substitution property</td>
</tr>
</tbody>
</table>
You can use the special angle pairs formed by parallel lines and a transversal to find missing angle measures.

**Problem**

If \( m\angle 1 = 100 \), what are the measures of \( \angle 2 \) through \( \angle 8 \)?

**Supplementary angles:**
\[
m\angle 2 = 180 - 100 \quad m\angle 2 = 80 \\
m\angle 4 = 180 - 100 \quad m\angle 4 = 80
\]

**Vertical angles:**
\[
m\angle 1 = m\angle 3 \quad m\angle 3 = 100
\]

**Alternate exterior angles:**
\[
m\angle 1 = m\angle 7 \quad m\angle 7 = 100
\]

**Alternate interior angles:**
\[
m\angle 3 = m\angle 5 \quad m\angle 5 = 100
\]

**Corresponding angles:**
\[
m\angle 2 = m\angle 6 \quad m\angle 6 = 80
\]

**Same-side interior angles:**
\[
m\angle 3 + m\angle 8 = 180 \quad m\angle 8 = 80
\]

**Problem**

What are the measures of the angles in the figure?

\[
(2x + 10) + (3x - 5) = 180 \quad \text{Same-Side Interior Angles Theorem}
\]

\[
5x + 5 = 180 \quad \text{Combine like terms.}
\]

\[
5x = 175 \quad \text{Subtract 5 from each side.}
\]

\[
x = 35 \quad \text{Divide each side by 5.}
\]

Find the measure of these angles by substitution.

\[
2x + 10 = 2 \cdot 35 + 10 = 80 \quad 3x - 5 = 3 \cdot 35 - 5 = 100
\]

\[
2x - 20 = 2 \cdot 35 - 20 = 50
\]

To find \( m\angle 1 \), use the Same-Side Interior Angles Theorem:

\[
50 + m\angle 1 = 180, \text{ so } m\angle 1 = 130
\]

**Exercises**

Find the value of \( x \). Then find the measure of each labeled angle.
3-3  Reteaching
Proving Lines Parallel

Special angle pairs result when a set of parallel lines is intersected by a transversal. The converses of the theorems and postulates in Lesson 3-2 can be used to prove that lines are parallel.

Postulate 3-2: Converse of Corresponding Angles Postulate
If \( \angle 1 \cong \angle 5 \), then \( a \parallel b \).

Theorem 3-4: Converse of the Alternate Interior Angles Theorem
If \( \angle 3 \cong \angle 6 \), then \( a \parallel b \).

Theorem 3-5: Converse of the Same-Side Interior Angles Theorem
If \( \angle 3 \) is supplementary to \( \angle 5 \), then \( a \parallel b \).

Theorem 3-6: Converse of the Alternate Exterior Angles Theorem
If \( \angle 2 \cong \angle 7 \), then \( a \parallel b \).

Problem
For what value of \( x \) is \( b \parallel c \)?
The given angles are alternate exterior angles. If they are congruent, then \( b \parallel c \).

\[
2x - 22 = 118 \\
2x = 140 \\
x = 70
\]

Exercises
Which lines or line segments are parallel? Justify your answers.

1. \( AB \parallel CD \) because the \( \cong \) are alt. ext. angles.
2. \( WX \parallel YZ \) because the \( \cong \) are alt. int. \( \triangle \).

Find the value of \( x \) for which \( g \parallel h \). Then find the measure of each labeled angle.

4. \( 126^\circ \) \( g \) \( 55; 126 \)
5. \( (2x + 6)^\circ \) \( g \) \( 7; 42; 138 \)
6. \( (4x - 18)^\circ \) \( g \) \( 24; 78; 78 \)
A flow proof is a way of writing a proof and a type of graphic organizer. Statements appear in boxes with the reasons written below. Arrows show the logical connection between the statements.

**Problem**

Write a flow proof for Theorem 3-1: If a transversal intersects two parallel lines, then alternate interior angles are congruent.

**Given:** \( \ell \parallel m \)

**Prove:** \( \angle 2 \equiv \angle 3 \)

```
Given \( \ell \parallel m \)

\( \angle 1 \equiv \angle 3 \)

Transitive Property of \( \equiv \)

Vertical angles are \( \equiv \).
```

**Exercises**

Complete a flow proof for each.

7. Complete the flow proof for Theorem 3-2 using the following steps. Then write the reasons for each step.
   a. \( \angle 2 \) and \( \angle 3 \) are supplementary.
   b. \( \angle 1 \equiv \angle 3 \)
   c. \( \ell \parallel m \)
   d. \( \angle 1 \) and \( \angle 2 \) are supplementary.

Theorem 3-2: If a transversal intersects two parallel lines, then same side interior angles are supplementary.

**Given:** \( \ell \parallel m \)

**Prove:** \( \angle 2 \) and \( \angle 3 \) are supplementary.

```
Given \( \ell \parallel m \)

\( \angle 1 \equiv \angle 3 \)

\( \angle 2 \) and \( \angle 3 \) are supplementary. Substitution

\( \angle 1 \) and \( \angle 2 \) are supplementary. linear pair
```

8. Write a flow proof for the following:

**Given:** \( \angle 2 \equiv \angle 3 \)

**Prove:** \( a \parallel b \)

```
Given \( \angle 2 \equiv \angle 3 \)

\( \angle 3 \equiv \angle 1 \)

Substitution

\( \angle 2 \equiv \angle 1 \)

Vertical \( \angle \) are \( \equiv \).

If \( \equiv \) corresponding \( \angle \), then lines are parallel.
```
You can use angle pairs to prove that lines are parallel. The postulates and theorems you learned are the basis for other theorems about parallel and perpendicular lines.

**Theorem 3-7:** Transitive Property of Parallel Lines

If two lines are parallel to the same line, then they are parallel to each other.

If \( a \parallel b \) and \( b \parallel c \), then \( a \parallel c \). Lines \( a, b, \) and \( c \) can be in different planes.

**Theorem 3-8:** If two lines are perpendicular to the same line, then those two lines are parallel to each other.

This is only true if all the lines are in the same plane.

If \( a \perp d \) and \( b \perp d \), then \( a \parallel b \).

**Theorem 3-9:** Perpendicular Transversal Theorem

If a line is perpendicular to one of two parallel lines, then it is also perpendicular to the other line.

This is only true if all the lines are in the same plane.

If \( a \parallel b \) and \( c \parallel d \), then \( b \perp d \), and \( c \perp d \).

**Exercises**

1. Complete this paragraph proof of Theorem 3-7.

   **Given:** \( d \parallel e \), \( e \parallel f \)

   **Prove:** \( d \parallel f \)

   **Proof:** Because it is given that \( d \parallel e \), then \( \angle 1 \) is supplementary to \( \angle 2 \) by the **Same-Side Int. Angles** Theorem. Because it is given that \( e \parallel f \), then \( \angle 2 \equiv \angle 3 \) by the **Corresponding Angles** Postulate. So, by substitution, \( \angle 1 \) is supplementary to \( \angle 3 \). By the **Converse of the Same-Side Int. Angles** Theorem, \( d \parallel f \).

2. Write a paragraph proof of Theorem 3-8.

   **Given:** \( t \perp n \), \( t \perp o \)

   **Prove:** \( n \parallel o \)

   *Given that \( t \perp n \) and \( t \perp o \), \( m\angle 1 = 90 \) and \( m\angle 2 = 90 \), by def. of perpendicular lines. Thus \( \angle 1 \equiv \angle 2 \). So, \( n \parallel o \) because of the Converse of the Corr. \( \Delta \) Post.
3-4  **Re教学 (continued)**
Parallel and Perpendicular Lines

**Problem**

A carpenter is building a cabinet. A decorative door will be set into an outer frame.

a. If the lines on the door are perpendicular to the top of the outer frame, what must be true about the lines?

b. The outer frame is made of four separate pieces of molding. Each piece has angled corners as shown. When the pieces are fitted together, will each set of sides be parallel? Explain.

c. According to Theorem 3-8, lines that are perpendicular to the same line are parallel to each other. So, since each line is perpendicular to the top of the outer frame, all the lines are parallel.

**Know**

- The angles for the top and bottom pieces are 35°.
- The angles for the sides are 55°.

**Need**

- Determine whether each set of sides will be parallel.

**Plan**

- Draw the pieces as fitted together to determine the measures of the new angles formed. Use this to decide if each set of sides will be parallel.

The new angle is the sum of the angles that come together. Since \(35° + 55° = 90°\), the pieces form right angles. Two lines that are perpendicular to the same line are parallel. So, each set of sides is parallel.

**Exercises**

3. An artist is building a mosaic. The mosaic consists of the repeating pattern shown at the right. What must be true of \(a\) and \(b\) to ensure that the sides of the mosaic are parallel? \(a = 50°\) and \(b = 25°\)

4. **Error Analysis**  A student says that according to Theorem 3-9, if \(AB \parallel CF\) and \(AD \perp \overline{AB}\), then \(CF \perp \overline{AB}\). Explain the student’s error.

\(AB\) and \(CF\) are in different planes.
Triangle Angle-Sum Theorem:
The measures of the angles in a triangle add up to 180.

**Problem**
In the diagram at the right, \( \triangle ACD \) is a right triangle.
What are \( m\angle 1 \) and \( m\angle 2 \)?

**Step 1**
\[
m\angle 1 + m\angle DAB = 90 \quad \text{Angle Addition Postulate}
\]
\[
m\angle 1 + 30 = 90 \quad \text{Substitution Property}
\]
\[
m\angle 1 = 60 \quad \text{Subtraction Property of Equality}
\]

**Step 2**
\[
m\angle 1 + m\angle 2 + m\angle ABC = 180 \quad \text{Triangle Angle-Sum Theorem}
\]
\[
60 + m\angle 2 + 60 = 180 \quad \text{Substitution Property}
\]
\[
m\angle 2 + 120 = 180 \quad \text{Addition Property of Equality}
\]
\[
m\angle 2 = 60 \quad \text{Subtraction Property of Equality}
\]

**Exercises**
Find \( m\angle 1 \).

1. [Diagram]
2. [Diagram]
3. [Diagram]
4. [Diagram]
5. [Diagram]
6. [Diagram]

**Algebra** Find the value of each variable.

7. [Diagram]
8. [Diagram]
9. [Diagram]
In the diagram at the right, $\angle 1$ is an exterior angle of the triangle. An exterior angle is an angle formed by one side of a polygon and an extension of an adjacent side.

For each exterior angle of a triangle, the two interior angles that are not next to it are its remote interior angles. In the diagram, $\angle 2$ and $\angle 3$ are remote interior angles to $\angle 1$.

The Exterior Angle Theorem states that the measure of an exterior angle is equal to the sum of its remote interior angles. So, $m\angle 1 = m\angle 2 + m\angle 3$.

**Problem**

What are the measures of the unknown angles?

\[
m\angle ABD + m\angle BDA + m\angle BAD = 180
\]
\[
45 + m\angle 1 + 31 = 180
\]
\[
m\angle 1 = 104
\]
\[
m\angle ABD + m\angle BAD = m\angle 2
\]
\[
45 + 31 = m\angle 2
\]
\[
76 = m\angle 2
\]

**Exercises**

What are the exterior angle and the remote interior angles for each triangle?

10. exterior: $\angle 4$
   interior: $\angle 1, \angle 2$

11. exterior: $\angle NOP$
   interior: $\angle OMN, \angle MNO$

12. exterior: $\angle JLM$
   interior: $\angle JKL, \angle LJK$

Find the measure of the exterior angle.

13. 115
14. 143
15. 117
**3-6 Reteaching**

**Constructing Parallel and Perpendicular Lines**

**Parallel Postulate**  Through a point not on a line, there is exactly one line parallel to the given line.

**Problem**

**Given:** Point $D$ not on $BC$  

**Construct:** $DJ$ parallel to $BC$

**Step 1**  Draw $BD$.

**Step 2**  With the compass tip on $B$, draw an arc that intersects $BD$ between $B$ and $D$. Label this intersection point $F$. Continue the arc to intersect $BC$ at point $G$.

**Step 3**  Without changing the compass setting, place the compass tip on $D$ and draw an arc that intersects $BD$ above $B$ and $D$. Label this intersection point $H$.

**Step 4**  Place the compass tip on $F$ and open or close the compass so it reaches $G$. Draw a short arc at $G$.

**Step 5**  Without changing the compass setting, place the compass tip on $H$ and draw an arc that intersects the first arc drawn from $H$. Label this intersection point $J$.

**Step 6**  Draw $DJ$, which is the required line parallel to $BC$. 
3-6 Reteaching (continued)
Constructing Parallel and Perpendicular Lines

Exercises

Construct a line parallel to line \( m \) and through point \( Y \).

1. 

2. 

3. 

Perpendicular Postulate  Through a point not on a line, there is exactly one line perpendicular to the given line.

Problem

Given: Point \( D \) not on \( \overrightarrow{BC} \)

Construct: a line perpendicular to \( \overrightarrow{BC} \) through \( D \)

Step 1  Construct an arc centered at \( D \) that intersects \( \overrightarrow{BC} \) at two points. Label those points \( G \) and \( H \).

Step 2  Construct two arcs of equal length centered at points \( G \) and \( H \).

Step 3  Construct the line through point \( D \) and the intersection of the arcs from Step 2.

Construct a line perpendicular to line \( n \) and through point \( X \).

4. 

5. 

6. 

Sample answers shown.
To find the slope $m$ of a line, divide the change in the $y$ values by the change in the $x$ values from one point to another. Slope is $\frac{\text{rise}}{\text{run}}$ or $\frac{\text{change in } y}{\text{change in } x}$.

**Problem**

What is the slope of the line through the points $(-5, 3)$ and $(4, 9)$?

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{9 - 3}{4 - (-5)} = \frac{6}{9} = \frac{2}{3}$$

**Exercises**

Find the slope of the line passing through the given points.

1. $(-5, 2), (1, 8)$
2. $(1, 8), (2, 4)$
3. $(-2, -3), (2, -4)$

If you know two points on a line, or if you know one point and the slope of a line, then you can find the equation of the line.

**Problem**

Write an equation of the line that contains the points $J(4, -5)$ and $K(-2, 1)$. Graph the line.

If you know two points on a line, first find the slope using

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$  

$$m = \frac{1 - (-5)}{-2 - 4} = \frac{6}{-6} = -1$$

Now you know two points and the slope of the line. Select one of the points to substitute for $(x_1, y_1)$. Then write the equation using the point-slope form $y - y_1 = m(x - x_1)$.

$$y - 1 = -1(x + 2)$$

Simplify within parentheses. You may leave your equation in this form or further simplify to find the slope-intercept form.

$$y - 1 = -x - 2$$

$$y = -x - 1$$

Answer: Either $y - 1 = -1(x + 2)$ or $y = -x - 1$ is acceptable.
Exercises

Graph each line.

4. \( y = \frac{1}{2}x - 4 \)  
5. \( y - 4 = \frac{1}{3}(x + 3) \)  
6. \( y - 3 = -6(x - 3) \)

If you know two points on a line, or if you know one point and the slope of a line, then you can find the equation of the line using the formula \( y - y_1 = m(x - x_1) \).

Use the given information to write an equation for each line.

7. slope \(-1\), \(y\)-intercept 6 \( y = -x + 6 \)  
8. slope \(\frac{4}{5}\), \(y\)-intercept \(-3\) \( y = \frac{4}{5}x - 3 \)

9. \( y = -3x - 8 \)  
10. \( y = -\frac{1}{4}x - \frac{13}{4} \)

11. passes through (7, \(-4\)) and (2, \(-2\)) \( y + 4 = -\frac{2}{5}(x - 7) \)  
12. passes through (3, 5) and (-6, 1) \( y - 5 = \frac{4}{9}(x - 3) \)

Graph each line.

13. \( y = 4 \)  
14. \( x = -4 \)  
15. \( y = -2 \)  
16. \( y - 7 = -2(x - 1) \) \( y = -2x + 9 \)  
17. \( y + 2 = \frac{1}{3}(x + 5) \) \( y = \frac{1}{3}x - \frac{1}{3} \)  
18. \( y + 5 = -\frac{3}{2}(x - 3) \) \( y = -\frac{3}{2}x + \frac{1}{2} \)
Reteaching
Slopes of Parallel and Perpendicular Lines

Remember that parallel lines are lines that are in the same plane that do not intersect, and perpendicular lines intersect at right angles.

Problem

Write an equation for the line that contains $G(4, -3)$ and is parallel to $\overline{EF}$: $\frac{1}{2}x + 2y = 6$. Write another equation for the line that contains $G$ and is perpendicular to $\overline{EF}$. Graph the three lines.

Step 1 Rewrite in slope-intercept form: $y = \frac{1}{4}x + 3$.

Step 2 Use point-slope form to write an equation for each line.

Parallel line: $m = \frac{1}{4}$ Perpendicular line: $m = -4$

$y - (-3) = \frac{1}{4}(x - 4)$ $y - (-3) = -4(x - 4)$

$y = \frac{1}{4}x - 4$ $y = -4x + 13$

Exercises

In Exercises 1 and 2, are lines $m_1$ and $m_2$ parallel? Explain.

1. Yes; both lines have a slope of 0.

2. Yes; both lines have a slope of 3.

Find the slope of a line (a) parallel to and (b) perpendicular to each line.

3. $y = 3x + 4$  
   ||: 3; ⊥: $-\frac{1}{3}$

4. $y = \frac{3}{5}x + 3\frac{4}{5}$  
   ||: $\frac{5}{3}$; ⊥: $-\frac{3}{5}$

5. $y - 3 = -4(x + 1)$  
   ||: -4; ⊥: $\frac{1}{4}$

6. $4x - 2y = 8$  
   ||: 2; ⊥: $-\frac{1}{2}$

Write an equation for the line parallel to the given line that contains point $C$.

7. $y = \frac{1}{2}x + 4$; $C(-4, -3)$  
   $y = \frac{1}{2}x - 1$

8. $y = -\frac{1}{2}x - 3$; $C(3, 1)$  
   $y = -\frac{1}{2}x + 2\frac{1}{2}$

9. $y = \frac{1}{3}x - 2$; $C(-4, 3)$  
   $y = \frac{1}{3}x + 4\frac{1}{3}$

10. $y = -2x - 4$; $C(3, 3)$  
    $y = -2x + 9$
Write an equation for the line perpendicular to the given line that contains \( P \).

11. \( P(5, 3); y = 4x \quad y = -\frac{1}{4}x + 4 \frac{1}{4} \)
12. \( P(2, 5); 3x + 4y = 1 \quad y = \frac{4}{3}x + 2 \frac{1}{3} \)
13. \( P(2, 6); 2x - y = 3 \quad y = -\frac{1}{2}x + 7 \)
14. \( P(2, 0); 2x - 3y = -9 \quad y = -\frac{3}{2}x + 3 \)

**Problem**

Given points \( J(-1, 4), K(2, 3), L(5, 4), \) and \( M(0, -3) \), are \( \overrightarrow{JK} \) and \( \overrightarrow{LM} \) parallel, perpendicular, or neither?

\[-\frac{1}{3} \neq \frac{7}{5} \]

Their slopes are not equal, so they are not parallel.

\[\frac{1}{3} \cdot \frac{7}{5} \neq -1 \]

The product of their slopes is not \(-1\), so they are not perpendicular.

**Exercises**

Tell whether \( \overrightarrow{JK} \) and \( \overrightarrow{LM} \) are parallel, perpendicular, or neither.

15. \( J(2, 0), K(-1, 3), L(0, 4), M(-1, 5) \)
   \[ m_{JK} = -1; m_{LM} = -1; \text{ parallel} \]
16. \( J(-4, -5), K(5, 1), L(6, 0), M(4, 3) \)
   \[ m_{JK} = \frac{2}{3}; m_{LM} = -\frac{3}{2}; \text{ perpendicular} \]
17. \( \overrightarrow{JK} : 6y + x = 7 \)
   \[ m_{JK} = -\frac{1}{6}; m_{LM} = -\frac{1}{5}; \text{ neither} \]
18. \( \overrightarrow{JK} : 3x + 2y = 5 \)
   \[ m_{JK} = -\frac{3}{2}; m_{LM} = -\frac{4}{5}; \text{ neither} \]
19. \( \overrightarrow{JK} : 2x - y = 1 \)
   \[ m_{JK} = 2; m_{LM} = -\frac{1}{2}; \text{ perpendicular} \]
20. \( \overrightarrow{JK} : y = \frac{1}{5}x + 2 \)
   \[ m_{JK} = \frac{1}{5}; m_{LM} = 5; \text{ neither} \]
21. \( \overrightarrow{JK} : 2y + \frac{1}{2}x = -2 \)
   \[ m_{JK} = -\frac{1}{4}; m_{LM} = -\frac{1}{4}; \text{ parallel} \]
22. \( \overrightarrow{JK} : y = -1 \)
   \[ m_{JK} = 0; m_{LM} = \text{ undefined}; \text{ perpendicular} \]

23. **Right Triangle** Verify that \( \triangle ABC \) is a right triangle for \( A(0, -4), B(3, -2), \) and \( C(-1, 4) \). Graph the triangle and explain your reasoning.

\( \triangle ABC \) is a right triangle because \( \overrightarrow{AB} \perp \overrightarrow{BC} \).

The slope of \( \overrightarrow{AB} \) is \( \frac{2}{3} \). The slope of \( \overrightarrow{BC} \) is \(-\frac{3}{2}\).
4-1 Reteaching

Congruent Figures

Given \(ABCD \cong QRST\), find corresponding parts using the names. Order matters.

For example, \(\triangle ABCD \cong \triangle QRS\)  This shows that \(\angle A\) corresponds to \(\angle Q\).
\[\frac{\overline{AB}}{\overline{QS}}\] Therefore, \(\angle A \cong \angle Q\).

For example, \(\triangle ABCD \cong \triangle QRS\)  This shows that \(\overline{BC}\) corresponds to \(\overline{RS}\).
\[\frac{\overline{BC}}{\overline{RS}}\] Therefore, \(\overline{BC} \cong \overline{RS}\).

Exercises

Find corresponding parts using the order of the letters in the names.

1. Identify the remaining three pairs of corresponding angles and sides between \(ABCD\) and \(QRST\) using the circle technique shown above.

   Angles: \(ABCD\)  \(ABCD\)  \(ABCD\)
   Sides: \(ABCD\)  \(ABCD\)  \(ABCD\)

\[\frac{\angle B}{\angle R}, \frac{\angle C}{\angle S}, \frac{\angle D}{\angle T}, \frac{\overline{AB}}{\overline{QR}}, \frac{\overline{CD}}{\overline{ST}}, \text{ and } \frac{\overline{DA}}{\overline{TQ}}\]

2. Which pair of corresponding sides is hardest to identify using this technique?
   
   Answers may vary. Sample: \(\overline{AD}\) and \(\overline{QT}\)

Find corresponding parts by redrawing figures.

3. The two congruent figures below at the left have been redrawn at the right. Why are the corresponding parts easier to identify in the drawing at the right?

   \[\frac{\overline{AB}}{\overline{BC}}\]

   \[\frac{\angle A}{\angle B}\]

   Answers may vary. Sample: The drawing at the right shows figures in same orientation.

4. Redraw the congruent polygons at the right in the same orientation. Identify all pairs of corresponding sides and angles. Check students' work. \(\angle A\) and \(\angle P\), \(\angle B\) and \(\angle Q\), \(\angle C\) and \(\angle R\), \(\angle D\) and \(\angle S\), \(\angle E\) and \(\angle T\), \(\overline{AB}\) and \(\overline{PQ}\), \(\overline{BC}\) and \(\overline{QR}\), \(\overline{CD}\) and \(\overline{RS}\), \(\overline{DE}\) and \(\overline{ST}\), and \(\overline{EA}\) and \(\overline{TP}\) all correspond.

5. \(MNOP \cong QRST\). Identify all pairs of congruent sides and angles.

   \(\angle M \cong \angle Q\), \(\angle N \cong \angle R\), \(\angle O \cong \angle S\), \(\angle P \cong \angle T\),

   \(\overline{MN} \cong \overline{QR}\), \(\overline{NO} \cong \overline{RS}\), \(\overline{OP} \cong \overline{ST}\), and \(\overline{PM} \cong \overline{TQ}\)
Given $\triangle ABC \cong \triangle DEF$, $m\angle A = 30$, and $m\angle E = 65$, what is $m\angle C$?

How might you solve this problem? Sketch both triangles, and put all the information on both diagrams.

$m\angle A = 30$; therefore, $m\angle D = 30$. How do you know? Because $\angle A$ and $\angle D$ are corresponding parts of congruent triangles.

**Exercises**

Work through the exercises below to solve the problem above.

6. What angle in $\triangle ABC$ has the same measure as $\angle E$? What is the measure of that angle? Add the information to your sketch of $\triangle ABC$.

   $\angle B; 65$

7. You know the measures of two angles in $\triangle ABC$. How can you find the measure of the third angle?

   **Answers may vary. Sample:** Use Triangle Angle-Sum Thm. Set sum of all three angles equal to 180.

8. What is $m\angle C$? How did you find your answer?

   $85$;

   **Answers may vary. Sample:** $m\angle C = 180 - (60 + 35)$

Before writing a proof, add the information implied by each given statement to your sketch. Then use your sketch to help you with Exercises 9–12.

**Add the information implied by each given statement.**

9. Given: $\angle A$ and $\angle C$ are right angles.

   $m\angle A = m\angle C = 90$, $\overline{DA} \perp \overline{AB}$ and $\overline{DC} \perp \overline{BC}$

10. Given: $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$.

    $ABCD$ is a parallelogram because it has opposite sides that are congruent.

11. Given: $\angle ADB \cong \angle CBD$.

    $\overline{AD} \parallel \overline{BC}$

12. Can you conclude that $\angle ABD \cong \angle CDB$ using the given information above?

    If so, how?

    **Yes; use the Third Angles Thm.**

13. How can you conclude that the third side of both triangles is congruent?

    The third side is shared by both triangles and is congruent by the Refl. Prop. of Congruence.
You can prove that triangles are congruent using the two postulates below.

**Postulate 4-1: Side-Side-Side (SSS) Postulate**

If all three sides of a triangle are congruent to all three sides of another triangle, then those two triangles are congruent.

If $\overline{JK} \cong \overline{XY}$, $\overline{KL} \cong \overline{YZ}$, and $\overline{JL} \cong \overline{XZ}$, then $\triangle JKL \cong \triangle XYZ$.

In a triangle, the angle formed by any two sides is called the *included angle* for those sides.

**Postulate 4-2: Side-Angle-Side (SAS) Postulate**

If two sides and the included angle of a triangle are congruent to two sides and the included angle of another triangle, then those two triangles are congruent.

If $\overline{PQ} \cong \overline{DE}$, $\overline{PR} \cong \overline{DF}$, and $\angle P \cong \angle D$, then $\triangle PQR \cong \triangle DEF$.

$\angle P$ is included by $\overline{QP}$ and $\overline{PR}$. $\angle D$ is included by $\overline{ED}$ and $\overline{DF}$.

**Exercises**

1. What other information do you need to prove $\triangle TRF \cong \triangle DFR$ by SAS? Explain. $\overline{TR} \cong TR$; by the Reflexive Property of Congruence, $\overline{RF} \cong FR$. It is given that $\angle TRF \cong \angle DFR$. These are the included angles for the corresponding congruent sides.

2. What other information do you need to prove $\triangle ABC \cong \triangle DEF$ by SAS? Explain. $\angle B \cong \angle E$; These are the included angles between the corresponding congruent sides.

3. **Developing Proof** Copy and complete the flow proof.

   **Given:** $\overline{DA} \cong \overline{MA}$, $\overline{AJ} \cong \overline{AZ}$

   **Prove:** $\triangle JDA \cong \triangle ZMA$

   $\overline{DA} \cong \overline{MA}$

   $\overline{AJ} \cong \overline{AZ}$

   $\angle JAD \cong \angle ZAM$

   $\overline{DA} \cong \overline{MA}$

   $\overline{AJ} \cong \overline{AZ}$

   $\angle JAD \cong \angle ZAM$

   Vertical $\triangle$ are $\cong$. 

   $\triangle JDA \cong \triangle ZMA$
Would you use SSS or SAS to prove the triangles congruent? If there is not enough information to prove the triangles congruent by SSS or SAS, write *not enough information*. Explain your answer.

4.

5.

6.

Not enough information; two pairs of corresponding sides are congruent, but the congruent angles are not the included angles.

Not enough information; you need to know if \( GC \equiv DY \).

Not enough information; only two pairs of corresponding sides are congruent. You need to know if \( AB \equiv XY \) or \( \angle Z \equiv \angle C \).

7. Given: \( PO \equiv SO \), \( O \) is the midpoint of \( NT \).

Prove: \( \triangle NOP \equiv \triangle TOS \)

Statements: 1) \( PO \equiv SO \); 2) \( O \) is the midpoint of \( NT \); 3) \( NO \equiv TO \); 4) \( \angle NOP \equiv \angle TOS \); 5) \( \triangle NOP \equiv \triangle TOS \);
Reasons: 1) Given; 2) Given; 3) Def. of midpoint; 4) Vert. \( \angle \) are \( \equiv \); 5) SAS

8. Given: \( HI \equiv HG \), \( FH \perp GI \)

Prove: \( \triangle FHI \equiv \triangle FHG \)

Statements: 1) \( FH \equiv FH \); 2) \( HI \equiv HG \), \( FH \perp GI \); 3) \( \angle FHG \) and \( \angle FHI \) are rt. \( \angle \); 4) \( \angle FHG \equiv \angle FHI \); 5) \( \triangle FHI \equiv \triangle FHG \);
Reasons: 1) Refl. Prop.; 2) Given; 3) Def. of perpendicular; 4) All rt. \( \angle \) are \( \equiv \); 5) SAS

9. A carpenter is building a support for a bird feeder. He wants the triangles on either side of the vertical post to be congruent. He measures and finds that \( AB \equiv DE \) and that \( AC \equiv DF \).

What would he need to measure to prove that the triangles are congruent using SAS? What would he need to measure to prove that they are congruent using SSS?

For SAS, he would need to determine if \( \angle BAC \equiv \angle EDF \); for SSS, he would need to determine if \( BC \equiv EF \).

10. An artist is drawing two triangles. She draws each so that two sides are 4 in. and 5 in. long and an angle is 55°. Are her triangles congruent? Explain.

Answers may vary. Sample: Maybe; if both the 55° angles are between the 4-in. and 5-in. sides, then the triangles are congruent by SAS.
4-3 Reteaching
Triangle Congruence by ASA and AAS

Problem
Can the ASA Postulate or the AAS Theorem be applied directly to prove the triangles congruent?

a. Because \( \angle RDE \) and \( \angle ADE \) are right angles, they are congruent. \( ED \cong ED \) by the Reflexive Property of \( \cong \), and it is given that \( \angle R \cong \angle A \). Therefore, \( \triangle RDE \cong \triangle ADE \) by the AAS Theorem.

b. It is given that \( CH \cong FH \) and \( \angle F \cong \angle C \). Because \( \angle CHE \) and \( \angle FHB \) are vertical angles, they are congruent. Therefore, \( \triangle CHE \cong \triangle FHB \) by the ASA Postulate.

Exercises
Indicate congruences.

1. Copy the top figure at the right. Mark the figure with the angle congruence and side congruence symbols that you would need to prove the triangles congruent by the ASA Postulate.

2. Copy the second figure shown. Mark the figure with the angle congruence and side congruence symbols that you would need to prove the triangles congruent by the AAS Theorem.

3. Draw and mark two triangles that are congruent by either the ASA Postulate or the AAS Theorem. Check students’ work.

What additional information would you need to prove each pair of triangles congruent by the stated postulate or theorem?

4. ASA Postulate
\( \angle ABD \cong \angle CBD \)

5. AAS Theorem
\( \angle JMK \cong \angle LKM \),
\( \angle JKM \cong \angle LMK \),
\( \angle JMK \cong \angle LMK \), or
\( \angle JKM \cong \angle LKM \)

6. ASA Postulate
\( \angle ZXY \cong \angle ZVU \)

7. AAS Theorem
\( \angle Y \cong \angle O \)

8. AAS Theorem
\( \angle P \cong \angle A \)

9. ASA Postulate
\( \angle CYL \cong \angle ALY \)
10. Provide the reason for each step in the two-column proof.

**Given:** \( TX \parallel VW, \overline{TU} \equiv \overline{VU}, \angle XTU \equiv \angle WVU, \angle UWV \) is a right angle.

**Prove:** \( \triangle TUX \equiv \triangle VUW \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( \angle UWV ) is a right angle.</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( VW \perp XW )</td>
<td>2) Definition of perpendicular lines</td>
</tr>
<tr>
<td>3) ( \overline{TX} \parallel \overline{VW} )</td>
<td>3) Given</td>
</tr>
<tr>
<td>4) ( \overline{TX} \perp XW )</td>
<td>4) Perpendicular Transversal Theorem</td>
</tr>
<tr>
<td>5) ( \angle UXT ) is a right angle.</td>
<td>5) Definition of perpendicular lines</td>
</tr>
<tr>
<td>6) ( \angle UWV \equiv \angle UXT )</td>
<td>6) All right angles are congruent.</td>
</tr>
<tr>
<td>7) ( \overline{TU} \equiv \overline{VU} )</td>
<td>7) Given</td>
</tr>
<tr>
<td>8) ( \angle XTU \equiv \angle WVU )</td>
<td>8) Given</td>
</tr>
<tr>
<td>9) ( \triangle TUX \equiv \triangle VUW )</td>
<td>9) AAS Theorem</td>
</tr>
</tbody>
</table>

11. Write a paragraph proof.

**Given:** \( WX \parallel ZY; WZ \parallel XY \)

**Prove:** \( \triangle WXY \equiv \triangle ZYW \)

It is given that \( WX \parallel ZY \) and \( WZ \parallel XY \), so \( \angle XWY \equiv \angle ZYW \) and \( \angle XYW \equiv \angle ZYW \), by the Alternate Interior \( \angle \) Thm. \( WY \equiv YW \) by the Reflexive Property of \( \equiv \). So, by ASA Post. \( \triangle WXY \equiv \triangle ZYW \).

12. Developing Proof

Complete the proof by filling in the blanks.

**Given:** \( \angle A \equiv \angle C, \angle 1 \equiv \angle 2 \)

**Prove:** \( \triangle ABD \equiv \triangle CDB \)

**Proof:** \( \angle A \equiv \angle C \) and \( \angle 1 \equiv \angle 2 \) are given. \( DB \equiv BD \) by \( \) \_\_.

So, \( \triangle ABD \equiv \triangle CDB \) by \( \) \_\_\_ \_ AAS

13. Write a paragraph proof.

**Given:** \( \angle 1 \equiv \angle 6, \angle 3 \equiv \angle 4, \overline{LP} \equiv \overline{OP} \)

**Prove:** \( \triangle LMP \equiv \triangle ONP \)

\( \angle 3 \equiv \angle 4 \) is given. Therefore, \( m\angle 3 = m\angle 4 \), by def. of \( \equiv \). Because \( \angle 2 \) and \( \angle 3 \) are linear pairs, and \( \angle 4 \) and \( \angle 5 \) are linear pairs, the pairs of angles are suppl. Therefore, \( \angle 2 \equiv \angle 5 \) by the Congruent Suppl. Thm. \( \angle 1 \equiv \angle 6 \) and \( \overline{LP} \equiv \overline{OP} \) are given, so \( \triangle LMP \equiv \triangle ONP \), by the AAS Thm.
If you can show that two triangles are congruent, then you can show that all the corresponding angles and sides of the triangles are congruent.

**Problem**

**Given:** \( AB \parallel DC, \angle B \cong \angle D \)

**Prove:** \( BC \cong DA \)

In this case you know that \( AB \parallel DC \). \( AC \) forms a transversal and creates a pair of alternate interior angles, \( \angle BAC \) and \( \angle DCA \).

You have two pairs of congruent angles, \( \angle BAC \cong \angle DCA \) and \( \angle B \cong \angle D \). Because you know that the shared side is congruent to itself, you can use AAS to show that the triangles are congruent. Then use the fact that corresponding parts are congruent to show that \( BC \cong DA \). Here is the proof:

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>1) ( AB \parallel DC )</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( \angle BAC \cong \angle DCA )</td>
<td>2) Alternate Interior Angles Theorem</td>
</tr>
<tr>
<td>3) ( \angle B \cong \angle D )</td>
<td>3) Given</td>
</tr>
<tr>
<td>4) ( AC \cong CA )</td>
<td>4) Reflexive Property of Congruence</td>
</tr>
<tr>
<td>5) ( \triangle ABC \cong \triangle CDA )</td>
<td>5) AAS</td>
</tr>
<tr>
<td>6) ( BC \cong DA )</td>
<td>6) CPCTC</td>
</tr>
</tbody>
</table>

**Exercises**

1. Write a two-column proof.

   **Given:** \( MN \cong MP, NO \cong PO \)

   **Prove:** \( \angle N \cong \angle P \)

<table>
<thead>
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<tbody>
<tr>
<td>1) ( MN \cong MP, NO \cong PO )</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( MO \cong MO )</td>
<td>2) Reflexive Property of ( \cong )</td>
</tr>
<tr>
<td>3) ( \triangle MNO \cong \triangle MPO )</td>
<td>3) SSS</td>
</tr>
<tr>
<td>4) ( \angle N \cong \angle P )</td>
<td>4) CPCTC</td>
</tr>
</tbody>
</table>
2. Write a two-column proof.

**Given:** \( PT \) is a median and an altitude of \( \triangle PRS \).

**Prove:** \( PT \) bisects \( \angle RPS \).

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>1) ( PT ) is a median of ( \triangle PRS ).</td>
<td>1) ( ? ) Given</td>
</tr>
<tr>
<td>2) ( ? ) ( T ) is the midpoint of ( RS ).</td>
<td>2) Definition of median</td>
</tr>
<tr>
<td>3) ( ? ) ( RT \equiv ST )</td>
<td>3) Definition of midpoint</td>
</tr>
<tr>
<td>4) ( PT ) is an altitude of ( \triangle PRS ).</td>
<td>4) ( ? ) Given</td>
</tr>
<tr>
<td>5) ( PT \perp RS )</td>
<td>5) ( ? ) Definition of altitude</td>
</tr>
<tr>
<td>6) ( \angle PTS ) and ( \angle PTR ) are right angles.</td>
<td>6) ( ? ) Definition of perpendicular</td>
</tr>
<tr>
<td>7) ( ? ) ( \angle PTS \equiv \angle PTR )</td>
<td>7) All right angles are congruent.</td>
</tr>
<tr>
<td>8) ( ? ) ( PT \equiv PT )</td>
<td>8) Reflexive Property of Congruence</td>
</tr>
<tr>
<td>9) ( ? ) ( \triangle PTS \equiv \triangle PTR )</td>
<td>9) SAS</td>
</tr>
<tr>
<td>10) ( ? ) ( \triangle TPS \equiv \triangle TPR )</td>
<td>10) ( ? ) CPCTC</td>
</tr>
<tr>
<td>11) ( ? ) ( PT ) bisects ( \angle RPS ).</td>
<td>11) ( ? ) Definition of angle bisector</td>
</tr>
</tbody>
</table>

3. Write a two-column proof.

**Given:** \( QK \equiv QA \); \( QB \) bisects \( \angle KQA \).

**Prove:** \( KB \equiv AB \)

<table>
<thead>
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</thead>
<tbody>
<tr>
<td>1) ( QK \equiv QA ); ( QB ) bisects ( \angle KQA ).</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( \angle KQB \equiv \angle AQB )</td>
<td>2) Def. of ( \angle ) bis.</td>
</tr>
<tr>
<td>3) ( BQ \equiv BQ )</td>
<td>3) Refl. Prop. of Congruence</td>
</tr>
<tr>
<td>4) ( \triangle KBQ \equiv \triangle ABQ )</td>
<td>4) SAS</td>
</tr>
<tr>
<td>5) ( KB \equiv AB )</td>
<td>5) CPCTC</td>
</tr>
</tbody>
</table>

4. Write a two-column proof.

**Given:** \( ON \) bisects \( \angle JOH \), \( \angle J \equiv \angle H \)

**Prove:** \( JN \equiv HN \)

<table>
<thead>
<tr>
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<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( ON ) bisects ( \angle JOH ), ( \angle J \equiv \angle H )</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) ( \angle JON \equiv \angle HON )</td>
<td>2) Def. of ( \angle ) bis.</td>
</tr>
<tr>
<td>3) ( ON \equiv ON )</td>
<td>3) Refl. Prop. of Congruence</td>
</tr>
<tr>
<td>4) ( \triangle JON \equiv \triangle HON )</td>
<td>4) AAS</td>
</tr>
<tr>
<td>5) ( JN \equiv HN )</td>
<td>5) CPCTC</td>
</tr>
</tbody>
</table>
Two special types of triangles are isosceles triangles and equilateral triangles.

An *isosceles* triangle is a triangle with two congruent sides. The base angles of an isosceles triangle are also congruent. An altitude drawn from the shorter base splits an isosceles triangle into two congruent right triangles.

An *equilateral* triangle is a triangle that has three congruent sides and three congruent angles. Each angle measures $60^\circ$.

You can use the special properties of isosceles and equilateral triangles to find or prove different information about a given figure.

Look at the figure at the right.

You should be able to see that one of the triangles is equilateral and one is isosceles.

**Problem**

What is $m \angle A$?

$\triangle ABC$ is isosceles because it has two base angles that are congruent. Because the sum of the measures of the angles of a triangle is 180, and $m \angle B = 40$, you can solve to find $m \angle A$.

\[
m \angle A + m \angle B + m \angle BEA = 180 \\
m \angle A + 40 + m \angle A = 180 \\
2m \angle A + 40 = 180 \\
2m \angle A = 140 \\
m \angle A = 70
\]

Substitution Property

Combine like terms.

Subtraction Property of Equality

Division Property of Equality

**Problem**

What is $FC$?

$\triangle CFG$ is equilateral because it has three congruent angles. 

$CG = (2 + 2) = 4$, and $CG = FG = FC$.

So, $FC = 4$. 
Problem

What is the value of $x$?

Because $x$ is the measure of an angle in an equilateral triangle, $x = 60$.

Problem

What is the value of $y$?

$$m\angle DCE + m\angle DEC + m\angle EDC = 180$$

There are $180^\circ$ in a triangle.

$$60 + 70 + y = 180$$

Substitution Property

$$y = 50$$

Subtraction Property of Equality

Exercises

Complete each statement. Explain why it is true.

1. $\angle EAB \cong ?$
   $\angle EBA$; base angles of an isosceles triangle are congruent.

2. $\angle BCD \cong ? \cong \angle DBC$
   $\angle CDB$; the angles of an equilateral triangle are congruent.

3. $FG \cong ? \cong DF$
   $\overline{GD}$; the sides of an equilateral triangle are congruent.

Determine the measure of the indicated angle.

4. $\angle ACB$ 60

5. $\angle DCE$ 65

6. $\angle BCD$ 55

Algebra  Find the value of $x$ and $y$.

7. $y = 35; 35$

8. $x = 65; 50$

9. Reasoning  An exterior angle of an isosceles triangle has a measure 140.
   Find two possible sets of measures for the angles of the triangle. 40, 40, 100; 40, 70, 70
Two right triangles are congruent if they have congruent hypotenuses and if they have one pair of congruent legs. This is the *Hypotenuse-Leg (HL) Theorem.*

\[ \triangle ABC \cong \triangle PQR \] because they are both right triangles, their hypotenuses are congruent \((AC \cong PR)\), and one pair of legs is congruent \((BC \cong QR)\).

**Problem**

How can you prove that two right triangles that have one pair of congruent legs and congruent hypotenuses are congruent (The Hypotenuse-Leg Theorem)?

Both of the triangles are right triangles.

\[ \angle B \text{ and } \angle E \text{ are right angles.} \]
\[ AB \cong DE \text{ and } AC \cong DF. \]

How can you prove that \(\triangle ABC \cong \triangle DEF\)?

Look at \(\triangle DEF\). Draw a ray starting at \(F\) that passes through \(E\). Mark a point \(X\) so that \(EX = BC\). Then draw \(DX\) to create \(\triangle DEX\).

See that \(EX \cong BC\). (You drew this.) \(\triangle DEX \cong \triangle ABC\). (All right angles are congruent.) \(DE \cong AB\). (This was given.) So, by SAS, \(\triangle ABC \cong \triangle DEX\).

\(DX \cong AC\) (by CPCTC) and \(AC \cong DF\). (This was given.) So, by the Transitive Property of Congruence, \(DX \cong DF\). Then, \(\triangle DEX \cong \triangle DEF\). (All right angles are congruent.) By the Isosceles Theorem, \(\angle X \cong \angle F\). So, by AAS, \(\triangle DEX \cong \triangle DEF\).

Therefore, by the Transitive Property of Congruence, \(\triangle ABC \cong \triangle DEF\).

**Problem**

Are the given triangles congruent by the Hypotenuse-Leg Theorem? If so, write the triangle congruence statement.

\[ \angle F \text{ and } \angle H \text{ are both right angles, so the triangles are both right.} \]
\[ GI \cong IG \text{ by the Reflexive Property and } FI \cong HG \text{ is given.} \]

So, \(\triangle FIG \cong \triangle HGI\).
Exercises

Determine if the given triangles are congruent by the Hypotenuse-Leg Theorem. If so, write the triangle congruence statement.

1. \( \triangle TVU \) is not congruent

2. \( \triangle RST \) is congruent, \( \triangle RST \equiv \triangle TSZ \)

3. \( \triangle LMN \equiv \triangle RVS \)

4. \( \triangle OPQ \equiv \triangle ZYX \)

Measure the hypotenuse and length of the legs of the given triangles with a ruler to determine if the triangles are congruent. If so, write the triangle congruence statement.

5. \( \triangle ABC \equiv \triangle CMA \)

6. \( \triangle EFG \equiv \triangle HIJ \)

7. Explain why \( \triangle LMN \equiv \triangle OMN \). Use the Hypotenuse-Leg Theorem.
   Because \( \angle NML \) and \( \angle NMO \) are right angles, both triangles are right triangles. It is given that their hypotenuses are congruent. Because they share a leg, one pair of corresponding legs is congruent. All criteria are met for the triangles to be congruent by the Hypotenuse-Leg Theorem.

8. Visualize \( \triangle ABC \) and \( \triangle DEF \), where \( AB = EF \) and \( CA = FD \). What else must be true about these two triangles to prove that the triangles are congruent using the Hypotenuse-Leg Theorem? Write a congruence statement. \( \angle B \) and \( \angle E \) are right angles, or \( \angle C \) and \( \angle D \) are right angles. \( \triangle ABC \equiv \triangle DEF \) or \( \triangle ABC \equiv \triangle FED \).
Sometimes you can prove one pair of triangles congruent and then use corresponding parts of those triangles to prove another pair congruent. Often the triangles overlap.

**Problem**

Given: \(AB \cong CB, \ AE \cong CD, \ \angle AED \cong \angle CDE\)

Prove: \(\triangle ABE \cong \triangle CBD\)

Think about a plan for the proof. Examine the triangles you are trying to prove congruent. Two pairs of sides are congruent. If the included angles, \(\angle A\) and \(\angle C\), were congruent, then the triangles would be congruent by SAS.

If the overlapping triangles \(\triangle AED\) and \(\triangle CDE\) were congruent, then the angles would be congruent by corresponding parts. When triangles overlap, sometimes it is easier to visualize if you redraw the triangles separately.

Now use the plan to write a proof.

**Given:** \(AB \cong CB, \ AE \cong CD, \ \angle AED \cong \angle CDE\)

**Prove:** \(\triangle ABE \cong \triangle CBD\)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) (AE \cong CD, \ \angle AED \cong \angle CDE)</td>
<td>1) Given</td>
</tr>
<tr>
<td>2) (ED \cong ED)</td>
<td>2) Reflexive Property of (\cong)</td>
</tr>
<tr>
<td>3) (\triangle AED \cong \triangle CDE)</td>
<td>3) SAS</td>
</tr>
<tr>
<td>4) (\angle A \cong \angle C)</td>
<td>4) CPCTC</td>
</tr>
<tr>
<td>5) (AB \cong CB)</td>
<td>5) Given</td>
</tr>
<tr>
<td>6) (\triangle ABE \cong \triangle CBD)</td>
<td>6) SAS</td>
</tr>
</tbody>
</table>
Separate and redraw the overlapping triangles. Identify the vertices.

1. \( \triangle GLJ \) and \( \triangle HJL \)
2. \( \triangle MRP \) and \( \triangle NQS \)
3. \( \triangle FED \) and \( \triangle CDE \)

Fill in the blanks for the two-column proof.

4. **Given**: \( \angle AEG \cong \angle AFD, \overline{AE} \cong \overline{AF}, \overline{GE} \cong \overline{FD} \)
   **Prove**: \( \triangle AFG \cong \triangle AED \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( \angle AEG \cong \angle AFD, \overline{AE} \cong \overline{AF}, \overline{GE} \cong \overline{FD} )</td>
<td>1) ? Given</td>
</tr>
<tr>
<td>2) ? ( \triangle AEG \cong \triangle AFD )</td>
<td>2) SAS</td>
</tr>
<tr>
<td>3) ( \overline{AG} \cong \overline{AD}, \angle G \cong \angle D )</td>
<td>3) ? CPCTC</td>
</tr>
<tr>
<td>4) ? ( \overline{GE} \cong \overline{FD} )</td>
<td>4) Given</td>
</tr>
<tr>
<td>5) ( GE = FD )</td>
<td>5) ? Def. of ( \cong )</td>
</tr>
<tr>
<td>6) ( GF + FE = GE, FE + ED = FD )</td>
<td>6) ? Seg. Addition Post.</td>
</tr>
<tr>
<td>7) ( GF + FE = FE + ED )</td>
<td>7) ? Substitution Property</td>
</tr>
<tr>
<td>8) ? ( GF = ED )</td>
<td>8) Subtr. Prop. of Equality</td>
</tr>
<tr>
<td>9) ? ( \triangle AFG \cong \triangle AED )</td>
<td>9) ? SAS</td>
</tr>
</tbody>
</table>

Use the plan to write a two-column proof.

5. **Given**: \( \angle PSR \) and \( \angle PQR \) are right angles, \( \angle QPR \cong \angle SRP \).
   **Prove**: \( \triangle STR \cong \triangle QTP \)

Plan for Proof:
Prove \( \triangle QPR \cong \triangle SRP \) by AAS. Then use CPCTC and vertical angles to prove \( \triangle STR \cong \triangle QTP \) by AAS.
5-1 Reteaching
Midsegments of Triangles

Connecting the midpoints of two sides of a triangle creates a segment called a midsegment of the triangle.

Point X is the midpoint of AB.
Point Y is the midpoint of BC.
So, XY is a midsegment of \(\triangle ABC\).

There is a special relationship between a midsegment and the side of the triangle that is not connected to the midsegment.

Triangle Midsegment Theorem
- The midsegment is parallel to the third side of the triangle.
- The length of the midsegment is half the length of the third side.

\(XY \parallel AC\) and \(XY = \frac{1}{2}AC\).

Connecting each pair of midpoints, you can see that a triangle has three midsegments.
\(XY, YZ,\) and \(ZX\) are all midsegments of \(\triangle ABC\).

Because Z is the midpoint of AC, \(XY = AZ = ZC = \frac{1}{2}AC\).

**Problem**

QR is a midsegment of \(\triangle MNO\).

What is the length of MO?

Start by writing an equation using the Triangle Midsegment Theorem.

\[
\frac{1}{2}MO = QR \\
MO = 2QR \\
= 2(20) \\
= 40
\]

So, \(MO = 40\).
5-1  
**Reteaching (continued)**

### Midsegments of Triangles

**Problem**

\( \overline{AB} \) is a midsegment of \( \triangle GEF \). What is the value of \( x \)?

\[
2AB = GF \\
2(2x) = 20 \\
4x = 20 \\
x = 5
\]

**Exercises**

Find the length of the indicated segment.

1. \( AC \) 30  
2. \( TU \) 13  
3. \( SU \) 5.3  
4. \( MO \) 4.4  
5. \( GH \) 30  
6. \( JK \) 9

**Algebra**  In each triangle, \( \overline{AB} \) is a midsegment. Find the value of \( x \).

7. \( \frac{3x}{5x + 7} \)  
8. \( \frac{3x + 15}{2x + 5} \)  
9. \( \frac{15x + 6.5}{10x} \)

10. \( \frac{2x}{x - 7} \)  
11. \( \frac{2x - 17}{x} \)  
12. \( \frac{x}{27} \)
5-2 Reteaching
Perpendicular and Angle Bisectors

Perpendicular Bisectors
There are two useful theorems to remember about perpendicular bisectors.

Perpendicular Bisector Theorem
If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

Converse of the Perpendicular Bisector Theorem
If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

Problem
What is the value of $x$?
Since $A$ is equidistant from the endpoints of the segment, it is on the perpendicular bisector of $EG$. So, $EF = GF$ and $x = 4$.

Exercises
Find the value of $x$.

1. $A$  
2. $N$  
3. $F$  
4. $E$  
5. $Q$  
6. $R$
5-2 Reteaching (continued)

Perpendicular and Angle Bisectors

Angle Bisectors

There are two useful theorems to remember about angle bisectors.

Angle Bisector Theorem

If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

Converse of the Angle Bisector Theorem

If a point in the interior of an angle is equidistant from the sides of an angle, then the point is on the angle bisector.

Problem

What is the value of $x$?

Because point $A$ is in the interior of the angle and it is equidistant from the sides of the angle, it is on the bisector of the angle.

$$\angle BCA \cong \angle ECA$$

$$x = 40$$

Exercises

Find the value of $x$.

7. 70

8. 4

9. 10
5-3 Reteaching
Bisectors in Triangles

The Circumcenter of a Triangle
If you construct the perpendicular bisectors of all three sides of a triangle, the constructed segments will all intersect at one point. This point of concurrency is known as the circumcenter of the triangle.

It is important to note that the circumcenter of a triangle can lie inside, on, or outside the triangle.

The circumcenter is equidistant from the three vertices. Because of this, you can construct a circle centered on the circumcenter that passes through the triangle’s vertices. This is called a circumscribed circle.

Problem
Find the circumcenter of right \( \triangle ABC \).

First construct perpendicular bisectors of the two legs, \( AB \) and \( AC \). These intersect at (2, 2), the circumcenter.

Notice that for a right triangle, the circumcenter is on the hypotenuse.

Exercises
Coordinate Geometry Find the circumcenter of each right triangle.

1. 
\[
\begin{array}{c}
\text{Coordinate Geometry} \\
\text{Find the circumcenter of } \triangle ABC.
\end{array}
\]

4. \( A(0, 0) \quad (5, 4) \)
\( B(0, 8) \)
\( C(10, 8) \)

5. \( A(-7, 3) \quad (1, -2) \)
\( B(9, 3) \)
\( C(-7, -7) \)

6. \( A(-5, 2) \quad (-1, 4) \)
\( B(3, 2) \)
\( C(3, 6) \)
The Incenter of a Triangle

If you construct angle bisectors at the three vertices of a triangle, the segments will intersect at one point. This point of concurrency where the angle bisectors intersect is known as the incenter of the triangle.

It is important to note that the incenter of a triangle will always lie inside the triangle.

The incenter is equidistant from the sides of the triangle. You can draw a circle centered on the incenter that just touches the three sides of the triangle. This is called an inscribed circle.

Problem

Find the value of $x$.

The angle bisectors intersect at $P$. The incenter $P$ is equidistant from the sides, so $SP = PT$. Therefore, $x = 9$.

Note that $PV$, the continuation of the angle bisector, is not the correct segment to use for the shortest distance from $P$ to $AC$.

Exercises

Find the value of $x$.

7. $x = 14$
8. $5x - 6$
9. $x = 6$
10. $4x - 7$
11. $4 + x$
12. $4x + 8$
13. $x + 6$
14. $2x$
15. $2x - 12$
16. $2x - 6$
A median of a triangle is a segment that runs from one vertex of the triangle to the midpoint of the opposite side. The point of concurrency of the medians is called the centroid.

The medians of $\triangle ABC$ are $AM$, $CX$, and $BL$.

The centroid is point $D$.

An altitude of a triangle is a segment that runs from one vertex perpendicular to the line that contains the opposite side. The orthocenter is the point of concurrency for the altitudes. An altitude may be inside or outside the triangle, or a side of the triangle.

The altitudes of $\triangle QRS$ are $QT$, $RU$, and $SN$.

The orthocenter is point $V$.

Determine whether $\overline{AB}$ is a median, an altitude, or neither.

1. $\overline{AB}$ altitude
2. $\overline{AB}$ median
3. $\overline{AB}$ altitude
4. $\overline{AB}$ neither
5. Name the centroid. $Z$
6. Name the orthocenter. $P$
5-4 Reteaching (continued)
Medians and Altitudes

The medians of a triangle intersect at a point two-thirds of the distance from a vertex to the opposite side. This is the Concurrency of Medians Theorem.

\( \overline{CJ} \) and \( \overline{AH} \) are medians of \( \triangle ABC \)
and point \( F \) is the centroid.

\[
CF = \frac{2}{3} CJ
\]

**Problem**

Point \( F \) is the centroid of \( \triangle ABC \). If \( CF = 30 \), what is \( CJ \)?

\[
CF = \frac{2}{3} CJ \\
30 = \frac{2}{3} \times CJ \\
\frac{3}{2} \times 30 = CJ \\
45 = CJ
\]

**Exercises**

In \( \triangle VYX \), the centroid is \( Z \). Use the diagram to solve the problems.

7. If \( XR = 24 \), find \( XZ \) and \( ZR \).  **16; 8**

8. If \( XZ = 44 \), find \( XR \) and \( ZR \).  **66; 22**

9. If \( VZ = 14 \), find \( VP \) and \( ZP \).  **21; 7**

10. If \( VP = 51 \), find \( VZ \) and \( ZP \).  **34; 17**

11. If \( ZO = 10 \), find \( YZ \) and \( YO \).  **20; 30**

12. If \( YO = 18 \), find \( YZ \) and \( ZO \).  **12; 6**

In Exercises 13–16, name each segment.

13. a median in \( \triangle DEF \)  **DL**

14. an altitude in \( \triangle DEF \)  **FR**

15. a median in \( \triangle EHF \)  **HL**

16. an altitude in \( \triangle HEK \)  **HK or KE**
5-5 Reteaching
Indirect Proof

In an indirect proof, you prove a statement or conclusion to be true by proving the opposite of the statement to be false.

There are three steps to writing an indirect proof.

**Step 1:** State as a temporary assumption the opposite (negation) of what you want to prove.

**Step 2:** Show that this temporary assumption leads to a contradiction.

**Step 3:** Conclude that the temporary assumption is false and that what you want to prove must be true.

**Problem**

**Given:** There are 13 dogs in a show; some are long-haired and the rest are short-haired. There are more long-haired than short-haired dogs.

**Prove:** There are at least seven long-haired dogs in the show.

**Step 1:** Assume that fewer than seven long-haired dogs are in the show.

**Step 2:** Let \( \ell \) be the number of long-haired dogs and \( s \) be the number of short-haired dogs. Because \( \ell + s = 13 \), \( s = 13 - \ell \). If \( \ell \) is less than 7, \( s \) is greater than or equal to 7. Therefore, \( s \) is greater than \( \ell \). This contradicts the statement that there are more long-haired than short-haired dogs.

**Step 3:** Therefore, there are at least seven long-haired dogs.

**Exercises**

Write the temporary assumption you would make as a first step in writing an indirect proof.

1. **Given:** an integer \( q \); **Prove:** \( q \) is a factor of 34. **Assume** \( q \) is not a factor of 34.

2. **Given:** \( \triangle XYZ \); **Prove:** \( XY + XZ > YZ \). **Assume** \( XY + XZ \leq YZ \).

3. **Given:** rectangle \( GHII \); **Prove:** \( m \angle G = 90 \). **Assume** \( m \angle G \neq 90 \).

4. **Given:** \( \overline{XY} \) and \( \overline{XM} \); **Prove:** \( XY = XM \) **Assume** \( XY \neq XM \).

Write a statement that contradicts the given statement.

5. Whitney lives in an apartment. **Whitney does not live in an apartment.**

6. Marc does not have three sisters. **Marc has three sisters.**

7. \( \angle 1 \) is a right angle. **\( \angle 1 \) is an acute angle.**

8. Lines \( m \) and \( h \) intersect. **Lines \( m \) and \( h \) do not intersect.**
5-5 Reteaching (continued)
Indirect Proof

Problem

Given: \( \angle A \) and \( \angle B \) are not complementary.
Prove: \( \angle C \) is not a right angle.

Step 1: Assume that \( \angle C \) is a right angle.

Step 2: If \( \angle C \) is a right angle, then by the Triangle Angle-Sum Theorem,
\[ m\angle A + m\angle B + 90 = 180. \]
So \( m\angle A + m\angle B = 90. \) Therefore, \( \angle A \) and \( \angle B \) are complementary. But \( \angle A \) and \( \angle B \) are not complementary.

Step 3: Therefore, \( \angle C \) is not a right angle.

Exercises

Complete the proofs.

9. Arrange the statements given at the right to complete the steps of the indirect proof.

Given: \( \overline{XY} \neq \overline{YZ} \)
Prove: \( \angle 1 \neq \angle 4 \)

Step 1: _____ B
Step 2: _____ D
Step 3: _____ F
Step 4: _____ E
Step 5: _____ A
Step 6: _____ C

A. But \( \overline{XY} \neq \overline{YZ} \).
B. Assume \( \angle 1 \equiv \angle 4 \).
C. Therefore, \( \angle 1 \neq \angle 4 \).
D. \( \angle 1 \) and \( \angle 2 \) are supplementary, and \( \angle 3 \) and \( \angle 4 \) are supplementary.
E. According to the Converse of the Isosceles Triangle Theorem, \( \overline{XY} = \overline{YZ} \) or \( \overline{XY} \equiv \overline{YZ} \).
F. If \( \angle 1 \equiv \angle 4 \), then by the Congruent Supplements Theorem, \( \angle 2 \equiv \angle 3 \).

10. Complete the steps below to write a convincing argument using indirect reasoning.

Given: \( \triangle DEF \) with \( \angle D \neq \angle F \)
Prove: \( \overline{EF} \neq \overline{DE} \)

Step 1: _____ Assume \( \overline{EF} \equiv \overline{DE} \).
Step 2: _____ If \( \overline{EF} \equiv \overline{DE} \), then by the Isosceles Triangle Theorem, \( \angle D \equiv \angle F \).
Step 3: _____ But \( \angle D \neq \angle F \).
Step 4: _____ Therefore, \( \overline{EF} \neq \overline{DE} \).
5-6 Reteaching
Inequalities in One Triangle

For any triangle, if two sides are not congruent, then the larger angle is opposite the longer side (Theorem 5-10). Conversely, if two angles are not congruent, then the longer side is opposite the larger angle (Theorem 5-11).

Problem

Use the triangle inequality theorems to answer the questions.

a. Which is the largest angle of \( \triangle ABC \)?

\( \overline{AB} \) is the longest side of \( \triangle ABC \). \( \angle C \) lies opposite \( \overline{AB} \).

\( \angle C \) is the largest angle of \( \triangle ABC \).

b. What is \( m\angle E \)? Which is the shortest side of \( \triangle DEF \)?

\[ m\angle D + m\angle E + m\angle F = 180 \text{ Triangle Angle-Sum Theorem} \]
\[ 30 + m\angle E + 90 = 180 \text{ Substitution} \]
\[ 120 + m\angle E = 180 \text{ Addition} \]
\[ m\angle E = 60 \text{ Subtraction Property of Equality} \]

\( \angle D \) is the smallest angle of \( \triangle DEF \). Because \( \overline{FE} \) lies opposite \( \angle D \), \( \overline{FE} \) is the shortest side of \( \triangle DEF \).

Exercises

1. Draw three triangles, one obtuse, one acute, and one right. Label the vertices. Exchange your triangles with a partner.
   a. Identify the longest and shortest sides of each triangle.
   b. Identify the largest and smallest angles of each triangle.
   c. Describe the relationship between the longest and shortest sides and the largest and smallest angles for each of your partner’s triangles.

Check students’ work. The longest side will be opposite the largest angle. The shortest side will be opposite the smallest angle.

Which are the largest and smallest angles of each triangle?

2. largest: \( \angle DEF \); smallest: \( \angle DFE \)
3. largest: \( \angle PQR \); smallest: \( \angle PRQ \)
4. largest: \( \angle ACB \); smallest: \( \angle CBA \)

Which are the longest and shortest sides of each triangle?

5. longest: \( \overline{DF} \); shortest: \( \overline{FE} \)
6. longest: \( \overline{PQ} \); shortest: \( \overline{RQ} \)
7. longest: \( \overline{SV} \); shortest: \( \overline{ST} \)
Inequalities in One Triangle

For any triangle, the sum of the lengths of any two sides is greater than the length of the third side. This is the Triangle Inequality Theorem.

\[
\begin{align*}
AB + BC &> AC \\
AC + BC &> AB \\
AB + AC &> BC
\end{align*}
\]

**Problem**

A. Can a triangle have side lengths 22, 33, and 25?

Compare the sum of two side lengths with the third side length.

\[
\begin{align*}
22 + 33 &> 25 \\
22 + 25 &> 33 \\
25 + 33 &> 22
\end{align*}
\]

A triangle can have these side lengths.

B. Can a triangle have side lengths 3, 7, and 11?

Compare the sum of two side lengths with the third side length.

\[
\begin{align*}
3 + 7 &< 11 \\
3 + 11 &> 7 \\
11 + 7 &> 3
\end{align*}
\]

A triangle cannot have these side lengths.

C. Two sides of a triangle are 11 and 12 ft long. What could be the length of the third side?

Set up inequalities using \(x\) to represent the length of the third side.

\[
\begin{align*}
x + 11 &> 12 \\
x + 12 &> 11 \\
11 + 12 &> x \\
x &> 1 \\
x &> -1 \\
23 &> x
\end{align*}
\]

The side length can be any value between 1 and 23 ft long.

**Exercises**

8. Can a triangle have side lengths 2, 3, and 7? \text{no}

9. Can a triangle have side lengths 12, 13, and 7? \text{yes}

10. Can a triangle have side lengths 6, 8, and 9? \text{yes}

11. Two sides of a triangle are 5 cm and 3 cm. What could be the length of the third side? \text{less than 8 cm and greater than 2 cm}

12. Two sides of a triangle are 15 ft and 12 ft. What could be the length of the third side? \text{less than 27 ft and greater than 3 ft}
5-7 Reteaching
Inequalities in Two Triangles

Consider \( \triangle ABC \) and \( \triangle XYZ \). If \( \overline{AB} \cong \overline{XY} \), \( \overline{BC} \cong \overline{YZ} \), and \( m\angle Y > m\angle B \), then \( \overline{XZ} \cong \overline{AC} \). This is the Hinge Theorem (SAS Inequality Theorem).

**Problem**

Which length is greater, \( GI \) or \( MN \)?

- Identify congruent sides: \( \overline{MO} \cong \overline{GH} \) and \( \overline{NO} \cong \overline{HI} \).
- Compare included angles: \( m\angle H > m\angle O \).
- By the Hinge Theorem, the side opposite the larger included angle is longer.

So, \( GI > MN \).

**Problem**

At which time is the distance between the tip of a clock’s hour hand and the tip of its minute hand greater, 3:00 or 3:10?

- Think of the hour hand and the minute hand as two sides of a triangle whose lengths never change, and the distance between the tips of the hands as the third side. 3:00 and 3:10 can then be represented as triangles with two pairs of congruent sides. The distance between the tips of the hands is the side of the triangle opposite the included angle.
- At 3:00, the measure of the angle formed by the hour hand and minute hand is 90°. At 3:10, the measure of the angle is less than 90°.

So, the distance between the tip of the hour hand and the tip of the minute hand is greater at 3:00.

**Exercises**

1. What is the inequality relationship between \( LP \) and \( XA \) in the figure at the right? \( XA > LP \)

2. At which time is the distance between the tip of a clock’s hour hand and the tip of its minute hand greater, 5:00 or 5:15? \( 5:00 \)
Consider \( \triangle LMN \) and \( \triangle PQR \). If \( LM > PQ \), \( MN > QR \), and \( PR > LN \), then \( m \angle Q > m \angle M \). This is the Converse of the Hinge Theorem (SSS Inequality Theorem).

**Problem**

\( TR > ZX \). What is the range of possible values for \( x \)?

The triangles have two pairs of congruent sides, because \( RS = XY \) and \( TS = ZY \). So, by the Converse of the Hinge Theorem, \( m \angle S > m \angle Y \).

Write an inequality:

\[
72 > 5x + 2 \quad \text{Converse of the Hinge Theorem}
\]
\[
70 > 5x \quad \text{Subtract 2 from each side.}
\]
\[
14 > x \quad \text{Divide each side by 5.}
\]

Write another inequality:

\[
m \angle Y > 0 \quad \text{The measure of an angle of a triangle is greater than 0.}
\]
\[
5x + 2 > 0 \quad \text{Substitute.}
\]
\[
5x > -2 \quad \text{Subtract 2 from each side.}
\]
\[
x > -\frac{2}{5} \quad \text{Divide each side by 5.}
\]

So, \( -\frac{2}{5} < x < 14 \).

**Exercises**

Find the range of possible values for each variable.

3.

\[
\frac{2}{3} < x < 20
\]

4.

\[
\frac{(2x - 4) + 86}{8} < x < 45
\]

5. **Reasoning** An equilateral triangle has sides of length 5, and an isosceles triangle has side lengths of 5, 5, and 4. Write an inequality for \( x \), the measure of the vertex angle of the isosceles triangle. \( 60 > x > 0 \)
**The Polygon Angle-Sum Theorems**

**Interior Angles of a Polygon**

The angles on the inside of a polygon are called *interior angles*.

**Polygon Angle-Sum Theorem:**

The sum of the measures of the angles of an \( n \)-gon is \((n - 2)180\). 

You can write this as a formula. This formula works for regular and irregular polygons.

\[
\text{Sum of angle measures} = (n - 2)180
\]

**Problem**

What is the sum of the measures of the angles in a hexagon?

There are six sides, so \( n = 6 \).

\[
\text{Sum of angle measures} = (n - 2)180
\]
\[
= (6 - 2)180 \quad \text{Substitute 6 for} \ n.
\]
\[
= 4(180) \quad \text{Subtract.}
\]
\[
= 720 \quad \text{Multiply.}
\]

The sum of the measures of the angles in a hexagon is 720.

You can use the formula to find the measure of one interior angle of a regular polygon if you know the number of sides.

**Problem**

What is the measure of each angle in a regular pentagon?

\[
\text{Sum of angle measures} = (n - 2)180
\]
\[
= (5 - 2)180 \quad \text{Substitute 5 for} \ n.
\]
\[
= 3(180) \quad \text{Subtract.}
\]
\[
= 540 \quad \text{Multiply.}
\]

Divide by the number of angles:

\[
\text{Measure of each angle} = \frac{540}{5}
\]
\[
= 108 \quad \text{Divide.}
\]

Each angle of a regular pentagon measures 108.
Exercises

Find the sum of the interior angles of each polygon.

1. quadrilateral \(360\)  
2. octagon \(1080\)  
3. 18-gon \(2880\)
4. decagon \(1440\)  
5. 12-gon \(1800\)  
6. 28-gon \(4680\)

Find the measure of an interior angle of each regular polygon. Round to the nearest tenth if necessary.

7. decagon \(144\)  
8. 12-gon \(150\)  
9. 16-gon \(157.5\)
10. 24-gon \(165\)  
11. 32-gon \(168.8\)  
12. 90-gon \(176\)

Exterior Angles of a Polygon

The exterior angles of a polygon are those formed by extending sides. There is one exterior angle at each vertex.

Polygon Exterior Angle-Sum Theorem:
The sum of the measures of the exterior angles of a polygon is 360.

A pentagon has five exterior angles. The sum of the measures of the exterior angles is always 360, so each exterior angle of a regular pentagon measures 72.

Exercises

Find the measure of an exterior angle for each regular polygon. Round to the nearest tenth if necessary.

13. octagon \(45\)  
14. 24-gon \(15\)  
15. 34-gon \(10.6\)
16. decagon \(36\)  
17. heptagon \(51.4\)  
18. hexagon \(60\)
19. 30-gon \(12\)  
20. 28-gon \(12.9\)  
21. 36-gon \(10\)
22. Draw a Diagram  A triangle has two congruent angles, and an exterior angle that measures 140. Find two possible sets of angle measures for the triangle. Draw a diagram for each. \(40, 40, 100; 40, 70, 70\)
## 6-2 Reteaching
Properties of Parallelograms

### Parallelograms
Remember, a parallelogram is a quadrilateral with both pairs of opposite sides parallel. Here are some attributes of a parallelogram:

- The opposite sides are congruent.
- The consecutive angles are supplementary.
- The opposite angles are congruent.
- The diagonals bisect each other.

You can use these attributes to solve problems about parallelograms.

### Problem
Find the value of $x$.

Because the consecutive angles are supplementary,

$$x + 60 = 180$$

$$x = 120$$

### Problem
Find the value of $x$.

Because opposite sides are congruent,

$$x + 7 = 15$$

$$x = 8$$

### Problem
Find the value of $x$ and $y$.

Because the diagonals bisect each other, $y = 3x$ and $4x = y + 3$.

$$4x = y + 3$$
$$4x = 3x + 3 \quad \text{Substitute for } y.$$  
$$x = 3 \quad \text{Subtraction Property of } =$$  
$$y = 3x \quad \text{Given}$$  
$$y = 3(3) \quad \text{Substitute for } x.$$  
$$y = 9 \quad \text{Simplify.}$$
6-2 Reteaching (continued)
Properties of Parallelograms

Exercises

Find the value of $x$ in each parallelogram.

1. \[ \text{ } \]
2. \[ \]
3. \[ \]
4. \[ \]
5. \[ \]
6. \[ \]
7. \[ \]
8. \[ \]
9. \[ \]
10. \[ \]
11. \[ \]

13. Writing Write a statement about the consecutive angles of a parallelogram.
Consecutive angles of a parallelogram are supplementary.

14. Writing Write a statement about the opposite angles of a parallelogram.
Opposite angles of a parallelogram are congruent.

15. Reasoning One angle of a parallelogram is 47. What are the measures of the other three angles in the parallelogram?
47, 133, and 133
Is a quadrilateral a parallelogram?

There are five ways that you can confirm that a quadrilateral is a parallelogram.

If both pairs of opposite sides are parallel, then the quadrilateral is a parallelogram.

If both pairs of opposite sides are congruent, then the quadrilateral is a parallelogram.

If both pairs of opposite angles are congruent, then the quadrilateral is a parallelogram.

If the diagonals bisect each other, then the quadrilateral is a parallelogram.

If one pair of sides is both congruent and parallel, then the quadrilateral is a parallelogram.

Exercises

Can you prove that the quadrilateral is a parallelogram based on the given information? Explain.

1. Yes; opposite sides
2. No; not enough information
3. Yes; 1 pair of sides and parallel
4. Yes; diagonals bisect each other
5. Yes; opposite angles
6. Yes; opposite sides
6-3 Reteaching (continued)

Proving That a Quadrilateral Is a Parallelogram

Determine whether the given information is sufficient to prove that quadrilateral WXYZ is a parallelogram.

7. WY bisects ZX no
8. WX || ZY; WZ ≡ XY no
9. VZ ≡ VX; WX ≡ YZ no
10. ∠VWZ ≡ ∠VYX; WZ ≡ XY yes

You can also use the requirements for a parallelogram to solve problems.

Problem

For what value of x and y must figure ABCD be a parallelogram?

In a parallelogram, the two pairs of opposite angles are congruent. So, in ABCD, you know that x = 2y and 5y + 54 = 4x. You can use these two expressions to solve for x and y.

Step 1: Solve for y.

5y + 54 = 4x
5y + 54 = 4(2y) Substituting 2y for x.
5y + 54 = 8y Simplifying.
54 = 3y Subtracting 5y from each side.
18 = y Dividing each side by 3.

Step 2: Solve for x.

x = 2y Opposite angles of a parallelogram are congruent.

x = 2(18) Substituting 18 for y.

x = 36 Simplifying.

For ABCD to be a parallelogram, x must be 36 and y must be 18.

Exercises

For what value of x must the quadrilateral be a parallelogram?

14. 15. 16.
Rhombuses, rectangles, and squares share some characteristics. But they also have some unique features.

A rhombus is a parallelogram with four congruent sides.

A rectangle is a parallelogram with four congruent angles. These angles are all right angles.

A square is a parallelogram with four congruent sides and four congruent angles. A square is both a rectangle and a rhombus. A square is the only type of rectangle that can also be a rhombus.

Here is a Venn diagram to help you see the relationships.

There are some special features for each type of figure.

**Rhombus:** The diagonals are perpendicular.
   The diagonals bisect a pair of opposite angles.

**Rectangles:** The diagonals are congruent.

**Squares:** The diagonals are perpendicular.
   The diagonals bisect a pair of opposite angles (forming two 45° angles at each vertex).
   The diagonals are congruent.

**Exercises**

Decide whether the parallelogram is a rhombus, a rectangle, or a square.

1. \( \square \) 2. \( \square \) 3. \( \square \) 4. \( \square \)

   \( \text{rectangle} \)        \( \text{rhombus} \)        \( \text{rhombus} \)        \( \text{square} \)
List the quadrilaterals that have the given property. Choose among parallelogram, rhombus, rectangle, and square.

5. Opposite angles are supplementary. rectangle, square
6. Consecutive sides are \( \cong \). rhombus, square
7. Consecutive sides are \( \perp \). rectangle, square
8. Consecutive angles are \( \cong \). rectangle, square

You can use the properties of rhombuses, rectangles, and squares to solve problems.

**Problem**

Determine the measure of the numbered angles in rhombus \( DEFG \).

\( \angle 1 \) is part of a bisected angle. \( m\angle DFG = 48 \), so \( m\angle 1 = 48 \).

Consecutive angles of a parallelogram are supplementary.

\( m\angle EFG = 48 + 48 = 96 \), so \( m\angle DGF = 180 - 96 = 84 \).

The diagonals bisect the vertex angle, so \( m\angle 2 = 84 \div 2 = 42 \).

**Exercises**

Determine the measure of the numbered angles in each rhombus.

9. \( \begin{array}{c}
1 \\
35^\circ
\end{array} \quad \begin{array}{c}
2 \\
35; 55
\end{array} \)

10. \( \begin{array}{c}
1 \\
12^\circ
\end{array} \quad \begin{array}{c}
2 \\
78; 90
\end{array} \)

Determine the measure of the numbered angles in each figure.

11. rectangle \( ABCD \)

\( \begin{array}{c}
1 \\
30^\circ
\end{array} \quad \begin{array}{c}
3 \\
60; 30; 60; 30
\end{array} \quad \begin{array}{c}
2 \\
60^\circ
\end{array} \)

12. square \( LMNO \)

\( \begin{array}{c}
1 \\
90; 90; 45
\end{array} \quad \begin{array}{c}
2 \\
90
\end{array} \quad \begin{array}{c}
3 \\
30
\end{array} \quad \begin{array}{c}
4 \\
90
\end{array} \)

**Algebra** \( TUVW \) is a rectangle. Find the value of \( x \) and the length of each diagonal.

13. \( TV = 3x \text{ and } UW = 5x - 10 \)

\( 5; 15 \)

14. \( TV = 2x - 4 \text{ and } UW = x + 10 \)

\( 14; 24; 24 \)

15. \( TV = 6x + 4 \text{ and } UW = 4x + 8 \)

\( 2; 16; 16 \)

16. \( TV = 7x + 6 \text{ and } UW = 9x - 18 \)

\( 12; 90; 90 \)

17. \( TV = 8x - 2 \text{ and } UW = 5x + 7 \)

\( 3; 22; 22 \)

18. \( TV = 10x - 4 \text{ and } UW = 3x + 24 \)

\( 4; 36; 36 \)
Reteaching
Conditions for Rhombuses, Rectangles, and Squares

A parallelogram is a rhombus if either of these conditions is met:

1) The diagonals of the parallelogram are perpendicular. (Theorem 6-16)

2) A diagonal of the parallelogram bisects a pair of opposite angles. (Theorem 6-17)

A parallelogram is a rectangle if the diagonals of the parallelogram are congruent.

Exercises

Classify each of the following parallelograms as a rhombus, a rectangle, or a square. For each, explain.

1. \(\overline{MO} \cong \overline{PN}\)  
   Rectangle; the diagonals are \(\cong\).

2. \(\overline{AC} \cong \overline{BD}\)  
   Square; the diagonals are \(\cong\) and \(\perp\).

3. \(\overline{DEFG}\)  
   Rhombus; the diagonals bisect opposite angles.

Use the properties of rhombuses and rectangles to solve problems.

Problem

For what value of \(x\) is \(\square DEFG\) a rhombus?

In a rhombus, diagonals bisect opposite angles.

So, \(m \angle GDF = m \angle EDF\).

\[
(4x + 10) = (5x + 6)
\]

Set angle measures equal to each other.

\[
10 = x + 6
\]

Subtract 4x from each side.

\[
4 = x
\]

Subtract 6 from each side.

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6-5 Reteaching (continued)
Conditions for Rhombuses, Rectangles, and Squares

Exercises

4. For what value of \( x \) is \( \square WXYZ \) a rhombus? \( 11 \)

\[ W \quad 22^\circ \quad 2x^\circ \quad X \]
\[ Z \]
\[ Y \]

5. \( SQ = 14 \). For what value of \( x \) is \( \square PQRS \) a rectangle?

Solve for \( PT \). Solve for \( PR \). \( 6; 7; 14 \)

\[ P \quad \sqrt{4x - 11} \quad T \quad \perp \quad Q \]
\[ S \]
\[ T \]
\[ R \]

6. For what value of \( x \) is \( \square RSTU \) a rhombus?

What is \( m \angle SRT \)? What is \( m \angle URS \)?

\( 48; 50; 100 \)

\[ R \quad (2x - 46)^\circ \quad S \]
\[ U \quad (x + 2)^\circ \quad T \]

7. \( LN = 54 \). For what value of \( x \) is \( \square LMNO \) a rectangle? \( 12 \)

\[ L \quad 2x^\circ \quad P \]
\[ M \quad 2x^\circ \quad N \]

8. Given: \( \square ABCD \), \( \overline{AC} \perp \overline{BD} \) at \( E \).

Prove: \( ABCD \) is a rhombus.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( AE \cong CE )</td>
<td>1) ( \overline{AC} \perp \overline{BD} ) at ( E )</td>
</tr>
<tr>
<td>2) ( AC \perp \overline{BD} ) at ( E )</td>
<td>Given</td>
</tr>
<tr>
<td>3) ( \angle AED ) and ( \angle CED ) are right angles.</td>
<td>Definition of perpendicular lines</td>
</tr>
<tr>
<td>4) ( \angle AED \cong \angle CED )</td>
<td>All right angles are congruent.</td>
</tr>
<tr>
<td>5) ( \overline{DE} \cong \overline{DE} )</td>
<td>Reflexive Property of Congruence</td>
</tr>
<tr>
<td>6) ( \triangle AED \cong \triangle CED )</td>
<td>SAS Postulate</td>
</tr>
<tr>
<td>7) ( \overline{AD} \cong \overline{CD} )</td>
<td>CPCTC</td>
</tr>
<tr>
<td>8) ( \overline{AB} \cong \overline{CD}, \overline{AD} \cong \overline{BC} )</td>
<td>Opposite sides of a ( \square ) are ( \cong ).</td>
</tr>
<tr>
<td>9) ( \overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA} )</td>
<td>Transitive Property of Congruence</td>
</tr>
<tr>
<td>10) ( ABCD ) is a rhombus.</td>
<td>Definition of rhombus</td>
</tr>
</tbody>
</table>
A trapezoid is a quadrilateral with exactly one pair of parallel sides. The two parallel sides are called bases. The two nonparallel sides are called legs.

A pair of base angles share a common base.
\( \angle 1 \) and \( \angle 2 \) are one pair of base angles.
\( \angle 3 \) and \( \angle 4 \) are a second pair of base angles.

In any trapezoid, the midsegment is parallel to the bases. The length of the midsegment is half the sum of the lengths of the bases.

\[
MN = \frac{1}{2}(QR + ZX)
\]

An isosceles trapezoid is a trapezoid in which the legs are congruent. An isosceles trapezoid has some special properties:

Each pair of base angles is congruent. The diagonals are congruent.

\[
AC = BD
\]

**Exercises**

1. In trapezoid \( LMNO \), what is the measure of \( \angle OLM \)? 70
   What is the measure of \( \angle LMN \)? 110

2. \( WXYZ \) is an isosceles trapezoid and \( WY = 12 \). What is \( XZ \)? 12

3. \( XZ \) is the midsegment of trapezoid \( EFGH \). If \( FG = 8 \) and \( EH = 12 \), what is \( XZ \)? 10
Trapezoids and Kites

A kite is a quadrilateral in which two pairs of consecutive sides are congruent and no opposite sides are congruent.

In a kite, the diagonals are perpendicular. The diagonals look like the crossbars in the frame of a typical kite that you fly.

Notice that the sides of a kite are the hypotenuses of four right triangles whose legs are formed by the diagonals.

**Problem**

Write a two-column proof to identify three pairs of congruent triangles in kite $FGHJ$.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $m\angle FKG = m\angle GKH = m\angle HKJ = m\angle JKF = 90\degree$</td>
<td>1) Theorem 6-22</td>
</tr>
<tr>
<td>2) $FG \cong FJ$</td>
<td>2) Given</td>
</tr>
<tr>
<td>3) $FK \cong FK$</td>
<td>3) Reflexive Property of Congruence</td>
</tr>
<tr>
<td>4) $\triangle FKG \cong \triangle FKJ$</td>
<td>4) HL Theorem</td>
</tr>
<tr>
<td>5) $JK \cong KG$</td>
<td>5) CPCTC</td>
</tr>
<tr>
<td>6) $KH \cong KH$</td>
<td>6) Reflexive Property of Congruence</td>
</tr>
<tr>
<td>7) $\triangle JKH \cong \triangle GKH$</td>
<td>7) SAS Postulate</td>
</tr>
<tr>
<td>8) $JH \cong GH$</td>
<td>8) Given</td>
</tr>
<tr>
<td>9) $FH \cong FH$</td>
<td>9) Reflexive Property of Congruence</td>
</tr>
<tr>
<td>10) $\triangle FJH \cong \triangle FGH$</td>
<td>10) SSS Postulate</td>
</tr>
</tbody>
</table>

So $\triangle FKG \cong \triangle FKJ$, $\triangle JKH \cong \triangle GKH$, and $\triangle FJH \cong \triangle FGH$.

**Exercises**

In kite $FGHJ$ in the problem, $m\angle JFK = 38\degree$ and $m\angle KGH = 63\degree$. Find the following angle and side measures.

4. $m\angle FKJ \quad 90\degree$
5. $m\angle FJK \quad 52\degree$
6. $m\angle FKG \quad 90\degree$
7. $m\angle KFG \quad 38\degree$
8. $m\angle FGK \quad 52\degree$
9. $m\angle GKH \quad 90\degree$
10. $m\angle KHG \quad 27\degree$
11. $m\angle KJH \quad 63\degree$
12. $m\angle JHK \quad 27\degree$
13. If $FG = 4.25$, what is $JF$? $4.25$
15. If $JK = 8.5$, what is $GJ$? $17$
Below are some formulas that can help you classify figures on a coordinate plane.

To determine if line segments that form sides or diagonals are congruent, use the Distance Formula:

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \]

In the figure at the right, the length of \( \overline{AB} \) is \( \sqrt{(5 - 1)^2 + (4 - 1)^2} = \sqrt{4^2 + 3^2} = 5 \).

In the figure above right the length of \( \overline{BC} \) is \( \sqrt{(5 - 0)^2 + (4 - 4)^2} = \sqrt{5^2 + 0^2} = 5 \).

So, \( \overline{AB} \cong \overline{BC} \). The figure is an isosceles triangle.

To find the midpoint of a side or diagonal, use the Midpoint Formula.

\[ M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]

In the figure above, the midpoint of \( \overline{AB} \) is \( \left( \frac{1 + 5}{2}, \frac{1 + 4}{2} \right) = \left( \frac{6}{2}, \frac{5}{2} \right) = (3, 2.5) \).

To determine whether line segments that form sides or diagonals are parallel or perpendicular, use the Slope Formula.

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

In the figure at the right, the slope of \( \overline{AB} \) is \( \frac{4 - 1}{5 - 1} = \frac{3}{4} \).

The slope of \( \overline{TS} \) is \( \frac{6 - 3}{5 - 1} = \frac{3}{4} \). The line segments are parallel.

Lines with equal slopes are parallel.

Lines with slopes that have a product of \(-1\) are perpendicular.

Exercises

1. How could you use the formulas to determine if a polygon on a coordinate plane is a rhombus? Answers may vary. Sample: Use the Distance Formula to prove that the sides are all the same length.

2. How could you use the formulas to determine if a trapezoid on a coordinate plane is isosceles? Answers may vary. Sample: Use the Distance Formula to prove that the non-parallel pair of sides are congruent.

3. How could you use the formulas to determine if a quadrilateral on a coordinate plane is a kite? Answers may vary. Sample: Use the Distance Formula to show that there are two pairs of adjacent congruent sides and no side is congruent to the side opposite it.
6-7 Repeating (continued)
Polygons in the Coordinate Plane

**Problem**

Is \( \triangle ABC \) scalene, isosceles, or equilateral?

Find the lengths of the sides using the Distance Formula.

\[
BA = \sqrt{(6)^2 + (1)^2} = \sqrt{36 + 1} = \sqrt{37}
\]
\[
BC = \sqrt{(2)^2 + (4)^2} = \sqrt{20}
\]
\[
CA = \sqrt{(4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5
\]

The sides are all different lengths. So, \( \triangle ABC \) is scalene.

**Problem**

Is quadrilateral \( GHIJ \) a parallelogram?

Find the slopes of the opposite sides.

slope of \( GH \) = \( \frac{4 - 3}{0 - (-3)} = \frac{1}{3} \); slope of \(JI\) = \( \frac{-1 - (-2)}{4 - 1} = \frac{1}{3}\)

slope of \( HI \) = \( \frac{-1 - 4}{4 - 0} = \frac{-5}{4} \); slope of \( GJ \) = \( \frac{-2 - 3}{1 - (-3)} = \frac{-5}{4}\)

So, \(JI \parallel GH\) and \(HI \parallel GJ\). Therefore, \(GHIJ\) is a parallelogram.

**Exercises**

\( \triangle JKL \) has vertices at \( J(-2, 4), K(1, 6), \) and \( L(4, 4). \)

4. Determine whether \( \triangle JKL \) is scalene, isosceles, or equilateral. Explain.
   
   Isosceles; use the Distance Formula to find that \(JK = KL = \sqrt{13}\), and \(JL = 6\).

5. Determine whether \( \triangle JKL \) is a right triangle. Explain.
   
   No; use the Slope Formula. The slope of \(JK\) = \( \frac{6 - 4}{1 - (-2)} = \frac{2}{3}\), the slope of \(KL\) = \( \frac{6 - 4}{1 - (-2)} = \frac{2}{3}\), and the slope of \(JL\) = \( \frac{4 - 4}{-2 - 4} = 0\), so no sides are perpendicular.

6. Trapezoid \( ABCD \) has vertices at \( A(2, 1), B(12, 1), C(9, 4), \) and \( D(5, 4). \) Which formula would help you find out if this trapezoid is isosceles? Is this an isosceles trapezoid? Explain.
   
   Distance Formula; yes; the distances \( AD \) and \( BC \) are both \( \sqrt{18} \) or \( 3\sqrt{2} \). Because these distances represent the legs, \(ABCD\) is isosceles.
You can use variables instead of integers to name the coordinates of a polygon in the coordinate plane.

**Problem**

Use the properties of each figure to find the missing coordinates.

**rhombus** $MNPQ$

$M$ is at the origin $(0, 0)$. Because diagonals of a rhombus bisect each other, $N$ has $x$-coordinate $\frac{a}{2}$. Because the $x$-axis is a horizontal line of symmetry for the rhombus, $Q$ has coordinates $(\frac{a}{2}, -b)$.

**square** $ABCD$

Because all sides are congruent, $D$ has coordinate $(0, x)$. Because all angles are right, $C$ has coordinates $(x, x)$.

**Exercises**

Use the properties of each figure to find the missing coordinates.

1. parallelogram $OPQR$
2. rhombus $XYZW$
3. square $QRST$

4. A quadrilateral has vertices at $(a, 0)$, $(-a, 0)$, $(0, a)$, and $(0, -a)$. Show that it is a square.
   
   **Sample:** Each side has a length of $a\sqrt{2}$, making the figure a rhombus. One pair of opposite sides has a slope of 1, and the other pair has a slope of $-1$. Because the product of the slopes is $-1$, the sides are perpendicular and the rhombus is a square.

5. A quadrilateral has vertices at $(a, 0)$, $(0, a + 1)$, $(-a, 0)$, and $(0, -a - 1)$.
   
   Show that it is a rhombus.
   
   **Each side has a length of $\sqrt{2a^2 + 2a + 1}$. Therefore, the figure is a rhombus.**

6. Isosceles trapezoid $ABCD$ has vertices $A(0, 0)$, $B(x, 0)$, and $D(k, m)$. Find the coordinates of $C$ in terms of $x$, $k$, and $m$. Assume $AB \parallel CD$. $C(x - k, m)$
You can use a coordinate proof to prove geometry theorems. You can use the Distance Formula, the Slope Formula, and the Midpoint Formula when writing coordinate proofs. With the Midpoint Formula, using multiples of two to name coordinates makes computation easier.

**Problem**

Plan a coordinate proof to show that the diagonals of a square are congruent.

Draw and label a square on a coordinate grid. In square $ABCD$, $AB = BC = CD = DA$. Draw in the diagonals, $AC$ and $BD$.

Prove that $AC = BD$. Use the Distance Formula.

\[
CA = \sqrt{(0 - a)^2 + (a - 0)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} \\
BD = \sqrt{(a - 0)^2 + (a - 0)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2}
\]

So, $CA = BD$. The diagonals of the square are congruent.

**Exercises**

7. How would you use a coordinate proof to prove that the diagonals of a square are perpendicular?
   *Answers may vary. Sample: Use the Slope Formula to prove that the product of the slopes of the diagonals is $-1$.*

8. How would you use a coordinate proof to prove that the diagonals of a rectangle are congruent?
   *Answers may vary. Sample: Use the Distance Formula to prove that the lengths of the diagonals are equal.*

9. How would you use a coordinate proof to prove that if the midpoints of the sides of a trapezoid are connected they will form a parallelogram?
   *Answers may vary. Sample: Use the Midpoint Formula to find the midpoints, then use the Slope Formula to show that opposite sides in the new figure have equal slopes.*

10. How would you use a coordinate proof to prove that the diagonals of a parallelogram bisect one another?
    *Answers may vary. Sample: Use the Midpoint Formula to show that the midpoints of the diagonals are the same point.*

11. Classify quadrilateral $ABCD$ with vertices $A(0, 0)$, $B(a, -b)$, $C(c, -b)$, $D(a + c, 0)$ as precisely as possible. Explain.
    *Isosceles trapezoid; one pair of sides parallel, other opposite pair of sides $\parallel$, and diagonals $\perp$*

12. Classify quadrilateral $FGHJ$ with vertices $F(a, 0)$, $G(a, 2c)$, $H(b, 2c)$, and $J(b, c)$ as precisely as possible. Explain.
    *Trapezoid; one pair of sides is parallel, the other pair of sides is not parallel.*
A coordinate proof can be used to prove geometric relationships. A coordinate proof uses variables to name coordinates of a figure on a coordinate plane.

**Problem**

Use coordinate geometry to prove that the diagonals of a rectangle are congruent.

\[
AC = \sqrt{(k - 0)^2 + (m - 0)^2} \\
\quad = \sqrt{k^2 + m^2} \\
BD = \sqrt{(0 - k)^2 + (m - 0)^2} \\
\quad = \sqrt{(-k)^2 + m^2} \\
\quad = \sqrt{k^2 + m^2} \\
\overline{AC} \cong \overline{BD}
\]

**Exercises**

Use coordinate geometry to prove each statement.

1. Diagonals of an isosceles trapezoid are congruent.

2. The line containing the midpoints of two sides of a triangle is parallel to the third side.

3. The segments joining the midpoints of a rectangle form a rhombus.

**Exercises Solutions**

1. **Diagonals of an isosceles trapezoid are congruent.**

   Use the Distance Formula to find the lengths of the diagonals. Each has a length of \(\sqrt{c^2 + (b + a)^2}\). So, the diagonals are congruent.

2. **The line containing the midpoints of two sides of a triangle is parallel to the third side.**

   You need to use the Midpoint Formula to find the midpoints of two sides, and the Slope Formula to show that the line connecting the midpoints is parallel to the third side. The midpoints are \(\left(\frac{b}{2}, \frac{c}{2}\right)\) and \(\left(\frac{b + a}{2}, \frac{c}{2}\right)\). The line connecting the midpoints has slope 0 and is therefore parallel to the third side.

3. **The segments joining the midpoints of a rectangle form a rhombus.**

   You need to find the midpoints using the Midpoint Formula and lengths of segments connecting the midpoints with the Distance Formula. The midpoints are \(\left(\frac{a}{2}, 0\right), \left(a, \frac{b}{2}\right), \left(\frac{a}{2}, b\right),\) and \(\left(0, \frac{b}{2}\right)\). The segments joining these points are congruent, as they each have a length of \(\frac{1}{2}\sqrt{a^2 + b^2}\).
The example used the Distance Formula to prove two line segments congruent. When planning a coordinate proof, write down the formulas that you will need to use, and write what you can prove using those formulas.

**Exercises**

State whether you can reach each conclusion below using coordinate methods. Give a reason for each answer.

4. \( AB = \frac{1}{2} CD \).
   - Yes; use the Distance Formula to show that \( AB = \frac{1}{2} CD \).

5. \( \triangle ABC \) is equilateral.
   - Yes; use the Distance Formula to show all sides are equal.

6. Quadrilateral \( ABCD \) is a square.
   - Yes; use the Distance Formula to show all sides are equal and the Slope Formula to show all sides are perpendicular if the product of the slopes of any two adjacent sides is \(-1\).

7. The diagonals of a quadrilateral form right angles.
   - Yes; use the Slope Formula to show the diagonals are \( \perp \) if the product of their slopes is \(-1\).

8. Quadrilateral \( ABCD \) is a trapezoid.
   - Yes; use the Slope Formula to show that one pair of sides is parallel because the slopes are equal.

9. \( \triangle ABC \) is a right triangle.
   - Yes; use the Slope Formula to show that the product of the slopes of two sides is \(-1\).

10. Quadrilateral \( ABCD \) is a kite.
    - Yes; use the Distance Formula to check that there are two pairs of adjacent sides of the same length, and that all four sides are not the same length.

11. The diagonals of a quadrilateral form angles that measure 30 and 150.
    - No; I do not have coordinate methods to measure angles.

12. \( m \angle D = 33 \)
    - No; I do not have coordinate methods to measure angles.

13. \( \triangle ABC \) is scalene.
    - Yes; use the Distance Formula to show that all sides have a different length.

14. The segments joining midpoints of an equilateral triangle form an equilateral triangle.
    - Yes; use the Midpoint Formula to find the midpoints of the sides and the distance formula to show that the measure of each side is equal in the new triangle.

15. Quadrilateral \( KLMN \) is an isosceles trapezoid.
    - Yes; use the Slope Formula to show that one pair of sides is parallel (have equal slopes), and use the Distance Formula to show that the legs are congruent.
7-1 Reteaching
Ratios and Proportions

Problem

About 15 of every 1000 light bulbs assembled at the Brite Lite Company are defective. If the Brite Lite Company assembles approximately 13,000 light bulbs each day, about how many are defective?

Set up a proportion to solve the problem. Let \( x \) represent the number of defective light bulbs per day.

\[
\frac{15}{1000} = \frac{x}{13,000}
\]

15(13,000) = 1000\( x \) Cross Products Property

195,000 = 1000\( x \) Simplify.

\[
\frac{195,000}{1000} = x
\]

Divide each side by 1000.

195 = \( x \) Solve for the variable.

About 195 of the 13,000 light bulbs assembled each day are defective.

Exercises

Use a proportion to solve each problem.

1. About 45 of every 300 apples picked at the Newbury Apple Orchard are rotten. If 3560 apples were picked one week, about how many apples were rotten? 534

2. A grocer orders 800 gal of milk each week. He throws out about 64 gal of spoiled milk each week. Of the 9600 gal of milk he ordered over three months, about how many gallons of spoiled milk were thrown out? 768

3. Seven of every 20 employees at V & B Bank Company are between the ages of 20 and 30. If there are 13,220 employees at V & B Bank Company, how many are between the ages of 20 and 30? 4627

4. About 56 of every 700 picture frames put together on an assembly line have broken pieces of glass. If 60,000 picture frames are assembled each month, about how many will have broken pieces of glass? 4800

Algebra Solve each proportion.

5. \( \frac{300}{1600} = \frac{x}{4800} \) 900
6. \( \frac{40}{140} = \frac{700}{x} \) 2450
7. \( \frac{x}{2000} = \frac{17}{400} \) 85
8. \( \frac{35}{x} = \frac{150}{2400} \) 560
9. \( \frac{x}{1040} = \frac{290}{5200} \) 58
10. \( \frac{x}{42,000} = \frac{87}{500} \) 7308
11. \( \frac{x}{380} = \frac{180}{5700} \) 12
12. \( \frac{1200}{90,000} = \frac{270}{x} \) 20,250
13. \( \frac{325}{x} = \frac{7306}{56,200} \) 2500
In a proportion, the products of terms that are diagonally across the equal sign from each other are the same. This is called the Cross Products Property because the products cross at the equal sign.

\[
\begin{array}{c}
\frac{a}{b} = \frac{c}{d} \\
\Rightarrow \quad b \times c = a \times d
\end{array}
\]

Proportions have other properties:

Property (1) \( \frac{a}{b} = \frac{c}{d} \) is equivalent to \( \frac{b}{a} = \frac{d}{c} \). Use reciprocals of the ratios.

Property (2) \( \frac{a}{b} = \frac{c}{d} \) is equivalent to \( \frac{a}{c} = \frac{b}{d} \). Switch \( b \) and \( c \) in the proportion.

Property (3) \( \frac{a}{b} = \frac{c}{d} \) is equivalent to \( \frac{a + b}{b} = \frac{c + d}{d} \). Add the denominator to the numerator.

**Problem**

How can you use the Cross Products Property to verify Property (3)?

\[
\frac{a}{b} = \frac{c}{d} \quad \text{is equivalent to} \quad ad = bc.
\]

\[
\frac{a + b}{b} = \frac{c + d}{d} \quad \text{is equivalent to} \quad (a + b)d = b(c + d).
\]

Cross Products Property

\[
ad + bd = bc + bd
\]

Distributive Property

\[
ad = bc
\]

Subtraction Property of Equality

So, \( \frac{a}{b} = \frac{c}{d} \) is equivalent to \( \frac{a + b}{b} = \frac{c + d}{d} \).

**Exercises**

Use the proportion \( \frac{x}{10} = \frac{2}{z} \). Complete each statement. Justify your answer.

14. \( \frac{x}{2} = \) \( \frac{10}{z} \), Prop. of Proportions (2)

15. \( \frac{10}{x} = \) \( \frac{z}{2} \), Prop. of Proportions (1)

16. \( \frac{x + 10}{10} = \) \( \frac{2 + z}{x} \), Prop. of Proportions (3)

17. The ratio of width to length of a rectangle is 7 : 10. The width of the rectangle is 91 cm. Write and solve a proportion to find the length. \( \frac{7}{10} = \frac{91}{x} \); 130 cm

18. The ratio of the two acute angles in a right triangle is 5 : 13. What is the measure of each angle in the right triangle? 25, 65, 90
7-2  Reteaching  
Similar Polygons

Similar polygons have corresponding angles that are congruent and corresponding sides that are proportional. An extended proportion can be written for the ratios of corresponding sides of similar polygons.

Problem

Are the quadrilaterals at the right similar? If so, write a similarity statement and an extended proportion.

Compare angles: \( \angle A \cong \angle X, \angle B \cong \angle Y, \angle C \cong \angle Z, \angle D \cong \angle W \).

Compare ratios of sides: \( \frac{AB}{XY} = \frac{6}{3} = 2 \), \( \frac{CD}{ZW} = \frac{9}{4.5} = 2 \), \( \frac{BC}{YZ} = \frac{8}{4} = 2 \), \( \frac{DA}{WX} = \frac{4}{2} = 2 \).

Because corresponding sides are proportional and corresponding angles are congruent, \( ABCD \sim XYZW \).

The extended proportion for the ratios of corresponding sides is:

\[
\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZW} = \frac{DA}{WX}
\]

Exercises

If the polygons are similar, write a similarity statement and the extended proportion for the ratios of corresponding sides. If the polygons are not similar, write not similar.

1. \( KML \sim QSR, \frac{KM}{QS} = \frac{ML}{SR} = \frac{LK}{RQ} \)

2. \( BCA \sim YZX, \frac{BC}{YX} = \frac{CA}{ZX} = \frac{AB}{XY} \)

3. not similar

4. not similar
7-2 Reteaching (continued)
Similar Polygons

**Problem**

\( \triangle RST \sim \triangle UVW \). What is the scale factor?

What is the value of \( x \)?

Identify corresponding sides: \( \overline{RT} \) corresponds to \( \overline{UW} \), \( \overline{TS} \) corresponds to \( \overline{WV} \), and \( \overline{SR} \) corresponds to \( \overline{VU} \).

\[
\frac{RT}{UW} = \frac{TS}{WV}
\]

Compare corresponding sides.

\[
\frac{4}{2} = \frac{7}{x}
\]

Substitute.

\( 4x = 14 \)

Cross Products Property

\( x = 3.5 \)

Divide each side by 4.

The scale factor is \( \frac{4}{2} = \frac{7}{3.5} = 2 \). The value of \( x \) is 3.5.

**Exercises**

Give the scale factor of the polygons. Find the value of \( x \). Round answers to the nearest tenth when necessary.

5. \( ABCD \sim NMPO \) \( 5 : 3; 3.6 \)

6. \( \triangle XYZ \sim \triangle EFD \) \( 3 : 2; 9.3 \)

7. \( LMNO \sim RQTS \) \( 10 : 7; 8.1 \)

8. \( OPQRST \sim GHJKLM \) \( 4 : 3; 12 \)
7-3 Reteaching
Proving Triangles Similar

Problem
Are the triangles similar? How do you know? Write a similarity statement.

Given: $\overline{DC} \parallel \overline{BA}$
Because $\overline{DC} \parallel \overline{BA}$, $\angle A$ and $\angle D$ are alternate interior angles and are therefore $\equiv$. The same is true for $\angle B$ and $\angle C$. So, by AA $\sim$ Postulate, $\triangle ABX \sim \triangle DCX$.

Exercises
Determine whether the triangles are similar. If so, write a similarity statement and name the postulate or theorem you used. If not, explain.

1. $\triangle ABC \sim \triangle ZYX$ by the AA $\sim$ Postulate.
2. Not similar; not all corresponding sides are proportional.
3. $\triangle QEU \sim \triangle SIO$ by the SSS $\sim$ Theorem.
4. $\triangle ABC \sim \triangle RST$ by the AA $\sim$ Postulate.
5. Not similar; the congruent angles are not corresponding.
6. $\triangle BAC \sim \triangle XQR$ by the SAS $\sim$ Postulate.

7. Are all equilateral triangles similar? Explain.
Yes; by the SSS $\sim$ Theorem or by the AA $\sim$ Postulate

8. Are all isosceles triangles similar? Explain.
No; the vertex angles may differ in measure.

9. Are all congruent triangles similar? Are all similar triangles congruent? Explain.
All congruent triangles are similar, with ratio 1 : 1. Similar triangles are not necessarily congruent.
10. Provide the reason for each step in the two-column proof.

Given: \( LM \perp MO \)
\( PN \perp MO \)

Prove: \( \triangle LMO \sim \triangle PNO \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( LM \perp MO ), ( PN \perp MO )</td>
<td>1) ? Given</td>
</tr>
<tr>
<td>2) ( \angle PNO ) and ( \angle LMO ) are right ( \Delta ).</td>
<td>2) ? Definition of a perpendicular line</td>
</tr>
<tr>
<td>3) ( \angle PNO \cong \angle LMO )</td>
<td>3) ? All right ( \Delta ) are ( \cong ).</td>
</tr>
<tr>
<td>4) ( \angle O \cong \angle O )</td>
<td>4) ? Reflexive Property of ( \cong )</td>
</tr>
<tr>
<td>5) ( \triangle LMO \sim \triangle PNO )</td>
<td>5) ? AA ( \sim ) Postulate</td>
</tr>
</tbody>
</table>

11. Developing Proof  Complete the proof by filling in the blanks.

Given: \( AB \parallel EF \), \( AC \parallel DF \)

Prove: \( \triangle ABC \sim \triangle FED \)

Proof: \( AB \parallel EF \) and \( AC \parallel DF \) are given. \( EB \) is a transversal by ?. Definition of a transversal
\( \angle E \cong \angle B \) by ?. Alt. Int. \( \triangle \) Thm.
Similarly, \( \angle EDF \cong \angle BCA \) by ?. Alt. Ext. \( \triangle \) Thm.
So, \( \triangle ABC \sim \triangle FED \) by ?. AA \( \sim \) Postulate

12. Write a paragraph proof.

Given: \( AD \) and \( EC \) intersect at \( B \).

Prove: \( \triangle ABE \sim \triangle DBC \)

Answers may vary. Sample:
\[
\frac{BD}{AB} = \frac{4}{12} = \frac{1}{3} = \frac{CB}{EB} = \frac{5}{15} = \frac{1}{3} = \frac{CD}{EA} = \frac{3}{10}. \quad \text{So},
\]
\[
\frac{BD}{CB} = \frac{CD}{AE}. \quad \text{Therefore, the sides of} \ \triangle ABE \ \text{and} \ \triangle DBC \ \text{are proportional and} \ \triangle ABE \sim \triangle DBC,
\]
by SSS \( \sim \) Theorem.
Theorem 7-3

If you draw an altitude from the right angle to the hypotenuse of a right triangle, you create three similar triangles. This is Theorem 7-3.

\( \triangle FGH \) is a right triangle with right \( \angle FGH \) and the altitude of the hypotenuse \( \overline{JG} \). The two triangles formed by the altitude are similar to each other and similar to the original triangle. So, \( \triangle FGH \sim \triangle FJG \sim \triangle GJH \).

Two corollaries to Theorem 7-3 relate the parts of the triangles formed by the altitude of the hypotenuse to each other by their geometric mean.

The geometric mean, \( x \), of any two positive numbers \( a \) and \( b \) can be found with the proportion \( \frac{a}{x} = \frac{x}{b} \).

**Problem**

What is the geometric mean of 8 and 12?

\[
\frac{8}{x} = \frac{x}{12} \\
x^2 = 96 \\
x = \sqrt{96} = \sqrt{16 \cdot 6} = 4\sqrt{6}
\]

The geometric mean of 8 and 12 is \( 4\sqrt{6} \).

**Corollary 1 to Theorem 7-3**

The altitude of the hypotenuse of a right triangle divides the hypotenuse into two segments. The length of the altitude is the geometric mean of these segments.

Since \( \overline{CD} \) is the altitude of right \( \triangle ABC \), it is the geometric mean of the segments of the hypotenuse \( \overline{AD} \) and \( \overline{DB} \):

\[
\frac{AD}{CD} = \frac{CD}{DB}.
\]
Corollary 2 to Theorem 7-3

The altitude of the hypotenuse of a right triangle divides the hypotenuse into two segments. The length of each leg of the original right triangle is the geometric mean of the length of the entire hypotenuse and the segment of the hypotenuse adjacent to the leg. To find the value of \(x\), you can write a proportion.

\[
\frac{\text{segment of hypotenuse}}{\text{adjacent leg}} = \frac{\text{adjacent leg}}{\text{hypotenuse}} \quad \frac{4}{8} = \frac{8}{4 + x} \quad \text{Corollary 2}
\]

\[
4(4 + x) = 64 \quad \text{Cross Products Property}
\]

\[
16 + 4x = 64 \quad \text{Simplify.}
\]

\[
4x = 48 \quad \text{Subtract 16 from each side.}
\]

\[
x = 12 \quad \text{Divide each side by 4.}
\]

Exercises

Write a similarity statement relating the three triangles in the diagram.

1. \(\triangle NOP \sim \triangle TNP \sim \triangle TON\)
2. \(\triangle FHG \sim \triangle HMG \sim \triangle FMH\)

Algebra Find the geometric mean of each pair of numbers.

3. 2 and 8 \(\sqrt{2^2} = 4\)
4. 4 and 6 \(\sqrt{4 \cdot 6} = 2\sqrt{6}\)
5. 8 and 10 \(\sqrt{8 \cdot 10} = 4\sqrt{5}\)
6. 25 and 4 \(\sqrt{25 \cdot 4} = 10\)

Use the figure to complete each proportion.

7. \(\frac{i}{f} = \frac{f}{k}\)
8. \(\frac{i}{j} = \frac{j}{h}\)
9. \(\frac{i}{f} = \frac{f}{k}\)

10. Error Analysis A classmate writes the proportion \(\frac{3}{5} = \frac{5}{3 + b}\) to find \(b\). Explain why the proportion is incorrect and provide the right answer.

The altitude is the geometric mean for the two segments of the hypotenuse, not for one segment and the entire hypotenuse: \(\frac{3}{5} = \frac{5}{b}\)
The Side-Splitter Theorem states the proportional relationship in a triangle in which a line is parallel to one side while intersecting the other two sides.

**Theorem 7-4: Side-Splitter Theorem**

In $\triangle ABC$, $GH \parallel AB$. $GH$ intersects $BC$ and $AC$. The segments of $BC$ and $AC$ are proportional: $\frac{AG}{GC} = \frac{BH}{HC}$.

The corollary to the Side-Splitter Theorem extends the proportion to three parallel lines intercepted by two transversals.

If $AB \parallel CD \parallel EF$, you can find $x$ using the proportion:

$$\frac{2}{7} = \frac{3}{x}$$

$2x = 21$  \hspace{1cm} \text{Cross Products Property}

$x = 10.5$  \hspace{1cm} \text{Solve for } x.

**Theorem 7-5: Triangle-Angle-Bisector Theorem**

When a ray bisects the angle of a triangle, it divides the opposite side into two segments that are proportional to the other two sides of the triangle.

In $\triangle DEF$, $EG$ bisects $\angle E$. The lengths of $DG$ and $GF$ are proportional to their adjacent sides $DE$ and $EF$: $\frac{DG}{DE} = \frac{GF}{EF}$.

To find the value of $x$, use the proportion $\frac{3}{5} = \frac{x}{8}$.

$6x = 24$

$x = 4$

**Exercises**

Use the figure at the right to complete each proportion.

1. $\frac{RQ}{MN} = \frac{SR}{LM}$
2. $\frac{NO}{QP} = \frac{LM}{SR}$
3. $\frac{MN}{RQ} = \frac{NO}{QP}$
4. $\frac{SQ}{LN} = \frac{RP}{MO}$

**Algebra**  Solve for $x$.

5. [Diagram]
6. [Diagram]
7. [Diagram]
7-5  Reteaching (continued)

Proportions in Triangles

**Algebra** Solve for $x$.

8. \[
\begin{array}{c}
3 \\
1
\end{array} \quad \frac{x}{1.5}
\]

9. \[
\begin{array}{c}
4 \\
2
\end{array} \quad \frac{x}{3}
\]

10. \[
\begin{array}{c}
6 \\
2
\end{array} \quad \frac{4}{x}
\]

11. \[
\begin{array}{c}
2x \\
x + 2
\end{array} \quad \frac{15}{12.5}
\]

12. \[
\begin{array}{c}
2x + 2 \\
2x + 2
\end{array} \quad \frac{8}{10.5}
\]

13. \[
\begin{array}{c}
12 \\
x - 1
\end{array} \quad \frac{9}{x - 2}
\]

   The bisector of $\angle A$ meets $BC$ at point $N$.
   Find $BN$ and $CN$. $BN = 3 \frac{1}{3}$, $CN = 4 \frac{4}{5}$

15. $\overline{XY} \parallel \overline{CA}$. Point $X$ lies on $BC$ such that $BX = 2$, and $Y$ is on $BA$.
   Find $BY$. $1.5$

16. **Error Analysis** A classmate says you can use the Corollary to the Side-Splitter Theorem to find the value of $x$. Explain what is wrong with your classmate’s statement.
   The corollary states that the segments on the transversal, not the segments on the parallel lines, are proportional.

17. An angle bisector of a triangle divides the opposite side of the triangle into segments 6 and 4 in. long. The side of the triangle adjacent to the 6-in. segment is 9 in. long. How long is the third side of the triangle? 6 in.

18. **Draw a Diagram** $\triangle GHI$ has angle bisector $\overline{GM}$, and $M$ is a point on $\overline{HI}$.
   $GH = 4$, $HM = 2$, $GI = 9$. Solve for $MI$. Use a drawing to help you find the answer. 4.5

19. The lengths of the sides of a triangle are 7 mm, 24 mm, and 25 mm. Find the lengths to the nearest tenth of the segments into which the bisector of each angle divides the opposite side.
   $5.6$ mm and $19.4$ mm; $3.4$ mm and $3.6$ mm; $5.3$ mm and $18.8$ mm
The Pythagorean Theorem can be used to find the length of a side of a right triangle.

Pythagorean Theorem: \( a^2 + b^2 = c^2 \), where \( a \) and \( b \) are the legs of a right triangle, and \( c \) is the hypotenuse.

**Problem**

What is the value of \( g \)? Leave your answer in simplest radical form.

Using the Pythagorean Theorem, substitute \( g \) and 9 for the legs and 13 for the hypotenuse.

\[
\begin{align*}
\quad & a^2 + b^2 = c^2 \\
\quad & g^2 + 9^2 = 13^2 \quad \text{Substitute.} \\
\quad & g^2 + 81 = 169 \quad \text{Simplify.} \\
\quad & g^2 = 88 \quad \text{Subtract 81 from each side.} \\
\quad & g = \sqrt{88} \quad \text{Take the square root.} \\
\quad & g = \sqrt{4(22)} \quad \text{Simplify.} \\
\quad & g = 2\sqrt{22} \\
\end{align*}
\]

The length of the leg, \( g \), is \( 2\sqrt{22} \).

**Exercises**

Identify the values of \( a \), \( b \), and \( c \). Write \( ? \) for unknown values. Then, find the missing side lengths. Leave your answers in simplest radical form.

1. \( a = 4, b = 8, c = ?; 4\sqrt{5} \)
2. \( a = 7, b = 7, c = ?; 7\sqrt{2} \)
3. \( a = 6, b = ?, c = 10; 8 \)
4. \( a = 3, b = ?, c = \sqrt{45}; 6 \)

5. A square has side length 9 in. What is the length of the longest line segment that can be drawn between any two points of the square? \( 9\sqrt{2} \) in.

6. Right \( \triangle ABC \) has legs of lengths 4 ft and 7 ft. What is the length of the triangle's hypotenuse? \( \sqrt{65} \) ft

7. Televisions are sold by the length of the diagonal across the screen. If a new 48-in. television screen is 42 in. wide, how tall is the screen to the nearest inch? 23 in.
Use Theorems 8-3 and 8-4 to determine whether a triangle is acute or obtuse.

Let $a$ and $b$ represent the shorter sides of a triangle and $c$ represent the longest side. 

If $a^2 + b^2 > c^2$, then the triangle is acute.
If $a^2 + b^2 < c^2$, then the triangle is obtuse.

**Problem**

A triangle has side lengths 6, 8, and 11. Is the triangle acute, obtuse, or right?

1. Identify $a$, $b$, and $c$.
   
   $a = 6$, $b = 8$, $c = 11$

2. Substitute to find $a^2 + b^2$.
   
   $a^2 + b^2 = 6^2 + 8^2$

3. Substitute to find $c^2$.
   
   $a^2 + b^2 = 36 + 64$

4. $c^2 = 11^2$

5. $c^2 = 121$

6. $100 < 121$

7. $a^2 + b^2 < c^2$, so the triangle is obtuse.

**Exercises**

The lengths of the sides of a triangle are given. Classify each triangle as acute, right, or obtuse.

8. 7, 9, 10 **acute**
9. 18, 16, 24 **acute**
10. 3, 5, $5\sqrt{2}$ **obtuse**

11. 10, 10, $10\sqrt{2}$ **right**
12. 8, 6, 10.5 **obtuse**
13. 7, $7\sqrt{3}$, 14 **right**

14. 22, 13, 23 **acute**
15. 17, 19, 26 **obtuse**
16. 21, 28, 35 **right**

17. A local park in the shape of a triangle is being redesigned. The fencing around the park is made of three sections. The lengths of the sections of fence are 27 m, 36 m, and 46 m. The designer of the park says that this triangle is a right triangle.

   Is he correct? Explain.

   **No:** the triangle is obtuse; $27^2 + 36^2 = 2025$ and $46^2 = 2116$. So, by the Pythagorean Inequality Theorems, this triangle is obtuse, because $a^2 + b^2 < c^2$.

18. Your neighbor’s yard is in the shape of a triangle, with dimensions 120 ft, 84 ft, and 85 ft. Is the yard an acute, obtuse, or a right triangle? Explain.

   **Obtuse:** $84^2 + 85^2 < 120^2$
In a $45^\circ$-$45^\circ$-$90^\circ$ triangle, the legs are the same length.

$$\text{hypotenuse} = \sqrt{2} \times \text{leg}$$

Problem

What is the value of the variable, $s$?

In a $45^\circ$-$45^\circ$-$90^\circ$ triangle, the hypotenuse is $\sqrt{2}$ times the length of the leg.

$$10 = s\sqrt{2}$$

Divide both sides by $\sqrt{2}$.

$$s = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Rationalize the denominator.

$$s = 5\sqrt{2}$$

In a $45^\circ$-$45^\circ$-$90^\circ$ triangle the length of the leg is $\frac{\sqrt{2}}{2} \times$ hypotenuse.

Exercises

Complete each exercise.

1. Draw a horizontal line segment on centimeter grid paper so that the endpoints are at the intersections of grid lines.

   Sample at the right. Check students’ work.

2. Use a protractor and a straightedge to construct a $45^\circ$-$45^\circ$-$90^\circ$ triangle.

   Check students’ work.

3. Use the $45^\circ$-$45^\circ$-$90^\circ$ Triangle Theorem to calculate the lengths of the legs.

   Round to the nearest tenth.

   Sample: hypotenuse = 9 cm; leg = 6.4 cm

4. Measure the lengths of the legs to the nearest tenth of a centimeter.

   Compare your calculated results and your measured results.

   They are the same.

Use the diagrams below each exercise to complete Exercises 5–7.

5. Find the length of the leg of the triangle. $12\sqrt{2}$

6. Find the length of the hypotenuse of the triangle. $7\sqrt{2}$

7. Find the length of the leg of the triangle. $8\sqrt{2}$
8-2 **Reteaching** (continued)  
Special Right Triangles

In a $30^\circ$-$60^\circ$-$90^\circ$ triangle, the longer leg is opposite the $60^\circ$ angle and the shorter leg is opposite the $30^\circ$ angle.

- longer leg $= \sqrt{3} \times$ shorter leg
- hypotenuse $= 2 \times$ shorter leg

**Problem**

Find the value of each variable.

$$5 = \sqrt{3} s$$

In a $30^\circ$-$60^\circ$-$90^\circ$ triangle the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.

$$\frac{5}{\sqrt{3}} = s$$

Divide both side by $\sqrt{3}$.

$$s = \frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

Rationalize the denominator.

The length of the hypotenuse is twice the length of the shorter leg.

$$t = 2\left(\frac{5\sqrt{3}}{3}\right) = \frac{10\sqrt{3}}{3}$$

**Exercises**

Complete each exercise.

8. Draw a horizontal line segment on centimeter grid paper so that the endpoints are at the intersections of grid lines.

9. Use a protractor and a straightedge to construct a $30^\circ$-$60^\circ$-$90^\circ$ triangle with your segment as one of its sides. Check students’ work.

10. Use the $30^\circ$-$60^\circ$-$90^\circ$ Triangle Theorem to calculate the lengths of the other two sides. Round to the nearest tenth. **Sample: 3.5 and 7**

11. Measure the lengths of the sides to the nearest tenth of a centimeter. **Sample: 3.5 and 7**

12. Compare your calculated results with your measured results. **Sample: They are approximately equal.**

13. Repeat the activity with a different segment. Check students’ work.

For Exercises 14–17, find the value of each variable.

14. $d = 2\sqrt{3}; e = 4$

15. $f = \frac{4\sqrt{3}}{3}; g = \frac{8\sqrt{3}}{3}$

16. $x = 3\sqrt{3}; y = 3$

17. $u = 6\sqrt{3}; v = 12$
8-3 Reteaching

Trigonometry

Use trigonometric ratios to find the length of a side of a right triangle.

\[
\sin A = \frac{\text{opposite}}{\text{hypotenuse}}
\]

\[
\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}
\]

\[
\tan A = \frac{\text{opposite}}{\text{adjacent}}
\]

**Problem**

What is the value of \( x \) to the nearest tenth?

First, identify the information given.

The angle measure is 29. The length of the side opposite the angle is \( x \). The length of the hypotenuse is 13.

\[
\sin 29^\circ = \frac{\text{opposite}}{\text{hypotenuse}}
\]

Use the sine ratio.

\[
\sin 29^\circ = \frac{x}{13}
\]

Substitute.

\[13(\sin 29^\circ) = x\]

Multiply by 13.

\[6.3 = x\]

Solve for \( x \) using a calculator.

**Exercises**

Find the value of \( t \) to the nearest tenth.

1. \[
\begin{array}{c}
\text{22} \\
\text{44^\circ} \\
\text{t}
\end{array}
\]

2. \[
\begin{array}{c}
\text{10.5} \\
\text{46^\circ} \\
\text{t}
\end{array}
\]

Find the missing lengths in each right triangle. Round your answers to the nearest tenth.

3. \[
\begin{array}{c}
\text{8} \\
\text{20^\circ} \\
\text{D}
\end{array}
\]

4. \[
\begin{array}{c}
\text{X} \\
\text{5} \\
\text{60^\circ} \\
\text{Z} \\
\text{8.7; 10}
\end{array}
\]

5. \[
\begin{array}{c}
\text{S} \\
\text{50^\circ} \\
\text{R} \\
\text{8.4; 13.1}
\end{array}
\]

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When you know the length of one or more sides in a right triangle and are looking for the angle measures of the triangle, you should use inverse trigonometric ratios.

\( \sin^{-1}(x) \) is the measure of the angle where \( \frac{\text{opposite}}{\text{hypotenuse}} = x \).

Similarly, \( \cos^{-1}(x) \) is the measure of the angle where \( \frac{\text{adjacent}}{\text{hypotenuse}} = x \), and

\( \tan^{-1}(x) \) is the measure of the angle where \( \frac{\text{opposite}}{\text{adjacent}} = x \).

**Problem**

Find the measure of \( \angle T \) to the nearest degree.

First, identify the information given. The length of the side \textit{adjacent} to the angle is 33. The length of the \textit{hypotenuse} is 55.

\[
\cos T = \frac{33}{55} = 0.6
\]

Use the cosine ratio.

\[
T = \cos^{-1}(0.6)
\]

Use the inverse of the cosine ratio.

\[
T = 53^\circ
\]

Use a calculator to solve.

The measure of \( \angle T \) is about 53.

**Exercises**

Find \( m \angle M \) to the nearest degree.

6. 

7. 

8. 

9. 

10. 

11. 

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## Angle of Elevation

Suppose you are looking up at an airplane. The angle formed by a horizontal line and your line of sight to the airplane is called the **angle of elevation**.

## Angle of Depression

Now suppose you are standing on a cliff and looking down at a river below. The line stretches horizontally from your point of view on the cliff. Your angle of sight to the river below forms an **angle of depression** with the horizontal line.

You can use your knowledge of trigonometric ratios to determine distances and lengths using angles of elevation and depression.

## Using the Angle of Elevation

### Problem

Suppose you are looking up at the top of a building. The angle formed by your line of sight and a horizontal line is 35°. You are standing 80 ft from the building and your eyes are 4 ft above the ground. How tall is the building, to the nearest foot?

Look at the diagram and think about what you know. You can see that a right triangle is formed by a horizontal line, your line of sight, and the building. You know an angle and one length.

Remember: \( \tan A = \frac{\text{opposite length}}{\text{adjacent length}} \)

Let the opposite length be \( x \).

\[
\tan 35 = \frac{x}{80}
\]

\[
80 \tan 35 = x
\]

\[
x \approx 56 \text{ ft}
\]

Your eyes are 4 ft above the ground, so add 4 to the value of \( x \) to find the total height of the building: 56 ft + 4 ft = 60 ft.
Using the Angle of Depression

Problem

Suppose you are a lifeguard looking down at a swimmer in a swimming pool. Your line of sight forms a 55° angle with a horizontal line. You are 10 ft up in your seat. How far is the swimmer from the base of the lifeguard stand?

Look at the diagram and think about what you know. You can see that a right triangle is formed by the horizon line, your line of sight, and a vertical distance that is the same as your height in the seat. You know an angle and one length.

Remember: \( \tan A = \frac{\text{opposite length}}{\text{adjacent length}} \)

Let the unknown side length be \( x \).

\[
\tan 55 = \frac{10}{x}
\]

\[
x = \frac{10}{\tan 55}
\]

\[
x = 7 \text{ ft}
\]

The swimmer is 7 ft from the base of the lifeguard stand.

Exercises

Find the value of \( x \). Round the lengths to the nearest tenth of a unit.

1. 
2. 
3. 
4. 

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8-5 Reteaching
Vectors

A vector is a quantity with size (magnitude) and direction. For example, you could represent a car traveling due north at 30 mi/h for 3 h with a vector on a coordinate grid.

The y-axis is the north-south direction. Draw the vector with its tail at the origin and its head at the point that represents the completion of the 3-h trip. The vector has a direction, due north, and a magnitude, 90 mi.

Because vectors have magnitude and direction, you can determine both of these values for a vector.

**Direction of a Vector** You can use a compass arrangement on the coordinate grid to describe a vector’s direction.

This vector is 30° south of east.

This vector is 40° east of north.

This vector is 20° north of west.

**Exercises**

Sketch a vector with the given direction.

1. 40° north of east
2. 30° east of south
3. 50° north of west

Use compass directions to describe the direction of each vector.

4. 20° south of west or 70° west of south
5. 38° east of north or 52° north of east
6. 29° south of east or 61° east of south
8-5 Reteaching (continued)

Vectors

Magnitude of a Vector

The magnitude of a vector is its length. You can use the distance formula to determine the length, or magnitude, of a vector.

Problem

What is the magnitude of each vector?

- The vector is from the origin to (4, 5).
  \[ \sqrt{(4 - 0)^2 + (5 - 0)^2} = \sqrt{41} \approx 6.4 \]
  The magnitude is 6.4.

- The vector is from the origin to (-3, -4).
  \[ \sqrt{(-3 - 0)^2 + (-4 - 0)^2} = \sqrt{25} = 5 \]
  The magnitude is 5.

Exercises

Find the magnitude of the given vector to the nearest tenth.

7. \( \langle 2, 3 \rangle \) 3.6
8. \( \langle -8, 2 \rangle \) 8.2
9. \( \langle -3, 3 \rangle \) 4.2

Find the magnitude and direction of each vector. Round to the nearest tenth.

10. \( \langle 2, 3 \rangle \) 5.7
11. \( \langle -2, 4 \rangle \) 4.5
12. \( \langle 1, -4 \rangle \) 4.1

13. \( \langle -12, 5 \rangle \) 13; 22.6° north of west or 67.4° west of north
14. \( \langle 15, -8 \rangle \) 17; 28.1° south of east or 61.9° east of south
15. \( \langle -7, -24 \rangle \) 25; 16.3° west of south or 73.7° south of west
9-1  
Reteaching  
Translations

A translation is a type of transformation. In a translation, a geometric figure changes position, but does not change shape or size. The original figure is called the preimage and the figure following transformation is the image.

The diagram at the right shows a translation in the coordinate plane. The preimage is \( \triangle ABC \). The image is \( \triangle A'B'C' \).

Each point of \( \triangle ABC \) has moved 5 units left and 2 units up. Moving left is in the negative \( x \) direction, and moving up is in the positive \( y \) direction. So, the rule for the translation is \((x, y) \rightarrow (x - 5, y + 2)\).

All translations are isometries because the image and the preimage are congruent. In this case, \( \triangle ABC \cong \triangle A'B'C' \).

**Problem**

What are the images of the vertices of \( WXYZ \) for the translation \((x, y) \rightarrow (x + 5, y - 1)\)? Graph the image of \( WXYZ \).

\[
\begin{align*}
W(-4, 1) &\rightarrow (-4 + 5, 1 - 1), \text{ or } W'(1, 0) \\
X(-4, 4) &\rightarrow (-4 + 5, 4 - 1), \text{ or } X'(1, 3) \\
Y(-1, 4) &\rightarrow (-1 + 5, 4 - 1), \text{ or } Y'(4, 3) \\
Z(-1, 1) &\rightarrow (-1 + 5, 1 - 1), \text{ or } Z'(4, 0)
\end{align*}
\]

**Exercises**

Use the rule to find the images of the vertices for the translation.

1. \( \triangle MNO (x, y) \rightarrow (x + 2, y - 3) \)  
2. square \( JKLM (x, y) \rightarrow (x - 1, y) \)

\[
\begin{align*}
M'(0, 1), N'(-1, -2), O'(1, -3) & \quad \text{and} \quad J'(-2, 2), K'(1, 2), L'(1, -1), M'(-2, -1)
\end{align*}
\]
Problem

What rule describes the translation of $ABCD$ to $A'B'C'D'$?

To get from $A$ to $A'$ (or from $B$ to $B'$ or $C$ to $C'$ or $D$ to $D'$), you move 8 units left and 7 units down. The rule that describes this translation is $(x, y) \rightarrow (x - 8, y - 7)$.

Exercises

- On graph paper, draw the $x$- and $y$-axes, and label Quadrants I–IV.
- Draw a quadrilateral in Quadrant I. Make sure that the vertices are on the intersection of grid lines.
- Trace the quadrilateral, and cut out the copy.
- Use the cutout to translate the figure to each of the other three quadrants.

Name the rule that describes each translation of your quadrilateral.

3. from Quadrant I to Quadrant II
   4. from Quadrant I to Quadrant III
   5. from Quadrant I to Quadrant IV
   6. from Quadrant II to Quadrant III
   7. from Quadrant III to Quadrant I

Refer to $ABCD$ in the problem above.

8. Give the image of the vertices of $ABCD$ under the translation $(x, y) \rightarrow (x - 2, y - 5)$. $A'(0, -3), B'(1, 1), C'(4, -1), D'(5, -4)$

9. Give the image of the vertices of $ABCD$ under the translation $(x, y) \rightarrow (x + 2, y - 4)$. $A'(4, -2), B'(5, 2), C'(8, 0), D'(9, -3)$

10. Give the image of the vertices of $ABCD$ under the translation $(x, y) \rightarrow (x + 1, y + 3)$. $A'(3, 5), B'(4, 9), C'(7, 7), D'(8, 4)$
**9-2 Reteaching**

**Reflections**

A *reflection* is a type of transformation in which a geometric figure is flipped over a *line of reflection*.

In a reflection, a *preimage* and an *image* have opposite orientations, but are the same shape and size. Because the preimage and image are congruent, a reflection is an *isometry*.

**Problem**

What are the reflection images of \(\triangle MNO\) across \(x-\) and \(y-\)axes? Give the coordinates of the vertices of the images.

From the graph you can see that the reflection image of \(\triangle MNO\) across the \(x\)-axis has vertices at \((2, -3)\), \((3, -7)\), and \((5, -4)\). The reflection image of \(\triangle MNO\) across the \(y\)-axis has vertices at \((-2, 3)\), \((-3, 7)\), and \((-5, 4)\).

**Exercises**

*Use a sheet of graph paper to complete Exercises 1–5.*

1. Draw and label the \(x\)- and \(y\)-axes on a sheet of graph paper. Check students’ work.
2. Draw a scalene triangle in one of the four quadrants. Make sure that the vertices are on the intersection of grid lines. Check students’ work.
3. Fold the paper along the axes. Check students’ work.
4. Cut out the triangle, and unfold the paper. Check students’ work.
5. Label the coordinates of the vertices of the reflection images across the \(x\)- and \(y\)-axes. Check students’ work.
Reteaching (continued)

Reflections

To graph a reflection image on a coordinate plane, graph the images of each vertex. Each vertex in the image must be the same distance from the line of reflection as the corresponding vertex in the preimage.

<table>
<thead>
<tr>
<th>Reflection across the x-axis:</th>
<th>Reflection across the y-axis:</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, y) \rightarrow (x, -y))</td>
<td>((x, y) \rightarrow (-x, y))</td>
</tr>
<tr>
<td>The (x)-coordinate does not change.</td>
<td>The (y)-coordinate does not change.</td>
</tr>
<tr>
<td>The (y)-coordinate tells the distance from the (x)-axis.</td>
<td>The (x)-coordinate tells the distance from the (y)-axis.</td>
</tr>
</tbody>
</table>

Problem

\(\triangle ABC\) has vertices at \(A(2, 4), B(6, 4),\) and \(C(3, 1)\). What is the image of \(\triangle ABC\) reflected over the \(x\)-axis?

Step 1: Graph \(A',\) the image of \(A.\) It is the same distance from the \(x\)-axis as \(A.\) The distance from the \(y\)-axis has not changed. The coordinates for \(A'\) are \((2, -4)\).

Step 2: Graph \(B'.\) It is the same distance from the \(x\)-axis as \(B.\) The distance from the \(y\)-axis has not changed. The coordinates for \(B'\) are \((6, -4)\).

Step 3: Graph \(C'.\) It is the same distance from the \(x\)-axis as \(C.\) The coordinates for \(C'\) are \((3, -1)\).

Each figure is reflected across the line indicated. Find the coordinates of the vertices for each image.

6. \(\triangle FGH\) with vertices \(F(-1, 3), G(-5, 1),\) and \(H(-3, 5)\) reflected across \(x\)-axis \(F'(-1, -3), G'(-5, -1), H'(-3, -5)\)

7. \(\triangle CDE\) with vertices \(C(2, 4), D(5, 2),\) and \(E(6, 3)\) reflected across \(x\)-axis \(C'(2, -4), D'(5, -2), E'(6, -3)\)

8. \(\triangle JKL\) with vertices \(J(-1, -5), K(-2, -3),\) and \(L(-4, -6)\) reflected across \(y\)-axis \(J'(1, -5), K'(2, -3), L'(4, -6)\)

9. Quadrilateral \(WXYZ\) with vertices \(W(-3, 4), X(-4, 6), Y(-2, 6),\) and \(Z(-1, 4)\) reflected across \(y\)-axis \(W'(3, 4), X'(4, 6), Y'(2, 6), Z'(1, 4)\)
A turning of a geometric figure about a point is a rotation. The center of rotation is the point about which the figure is turned. The number of degrees the figure turns is the angle of rotation. (In this chapter, rotations are counterclockwise unless otherwise noted.)

A rotation is an isometry. The image and preimage are congruent.

\[ ABCD \text{ is rotated about } Z. \text{ ABCD is the preimage and } A'B'C'D' \text{ is the image. The center of rotation is point } Z. \text{ The angle of rotation is } 82^\circ. \]

A composition of rotations is two or more rotations in combination. If the center of rotation is the same, the measures for the angles of rotation can be added to find the total rotation of the combination.

**Exercises**

Complete the following steps to draw the image of \( \triangle XYZ \) under an 80° rotation about point \( T \).

1. Draw \( \angle XTX' \) so that \( m\angle XTX' = 80 \) and \( \overline{TX} \cong \overline{TX'} \).

2. Draw \( \angle ZTZ' \) so that \( m\angle ZTZ' = 80 \) and \( \overline{TZ} \cong \overline{TZ'} \).

3. Draw \( \angle YTY' \) so that \( m\angle YTY' = 80 \) and \( \overline{TY} \cong \overline{TY'} \).

4. Draw \( \overline{X'Z'}, \overline{X'Y'}, \) and \( \overline{Y'Z'} \) to complete \( \triangle X'Y'Z' \).

Copy \( \triangle XYZ \) to complete Exercises 5–7.

5. Draw the image of \( \triangle XYZ \) under a 120° rotation about \( T \).

6. Draw a point \( S \) inside \( \triangle XYZ \). Draw the image of \( \triangle XYZ \) under a 135° rotation about \( S \).

7. Draw the image of \( \triangle XYZ \) under a 90° rotation about \( Y \).
In a regular polygon, the center is the same distance from every vertex. Regular polygons can be divided into a number of congruent triangles. The number of triangles is the same as the number of sides of the polygon. The measure of each central angle (formed by one vertex, the center, and an adjacent vertex) is \( \frac{360}{\text{number of congruent triangles}} \).

In regular hexagon \( MNOPQR \), the center and the vertices can be used to divide the hexagon into six congruent triangles. The measure of each central angle is \( \frac{360}{6} \), or 60.

**Problem**

In regular pentagon \( QRSTU \), what is the image of point \( Q \) for a rotation of 144° about point \( Z \)?

First find the measure of the central angle of a regular pentagon.

\[
m\angle RZS = \frac{360}{5} = 72
\]

When \( Q \) rotates 72°, it moves one vertex counterclockwise. When \( Q \) rotates 144°, it moves two vertices counterclockwise. So, for a rotation of 144°, the image of \( Q \) is point \( T \).

**Exercises**

Point \( Z \) is the center of regular pentagon \( QRSTU \). Find the image of the given point or segment for the given rotation.

8. 216° rotation of \( S \) about \( Z \) \( U \)
9. 144° rotation of \( TU \) about \( Z \) \( R5 \)
10. 360° rotation of \( Q \) about \( Z \) \( Q \)
11. 288° rotation of \( R \) about \( Z \) \( S \)

12. What is the measure of the angle of rotation that maps \( T \) onto \( U \)?
   (Hint: How many vertices away from \( T \) is \( U \), counterclockwise?) \( 288° \)

13. What is the measure of the angle of rotation for the regular hexagon \( ABCDEF \) that maps \( A \) onto \( C \)? \( 120° \) or \( 240° \)

14. What is the measure of the angle of rotation for the regular octagon \( DEFGHIJK \) that maps \( F \) onto \( K \)? \( 135° \) or \( 225° \)
A figure with symmetry can be reflected onto itself or rotated onto itself.

**Line Symmetry**

Line symmetry is also called reflectional symmetry. If a figure has line symmetry, it has a line of symmetry that divides the figure into two congruent halves. A figure may have one or more than one line of symmetry.

The heart has line symmetry. The dashed line is its one line of symmetry.

**Rotational and Point Symmetry**

If a figure has rotational symmetry, there is a rotation of one-half turn (180°) or less for which the figure is its own image.

The star at the right has rotational symmetry. The smallest angle needed for the star to rotate onto itself is 45°. This is the angle of rotation.

A figure has point symmetry if a 180° rotation maps the figure onto itself. The letter S has point symmetry.

**Three-Dimensional Symmetry**

The regular hexagonal prism has reflectional symmetry in a plane; the plane divides the prism into two congruent halves.

The regular hexagonal prism also has rotational symmetry. Any rotation that is a multiple of 60° around the dashed line will map the prism onto itself.
Consider the following types of symmetry: rotational, point, and (line) reflectional.

**Problem**

What types of symmetry does the flag have?

The flag has four lines of symmetry shown by the dotted lines. It has $90^\circ$ rotational symmetry and point symmetry about its center.

**Exercises**

Describe the symmetries in each flag.

1. [Flag of Israel]
   - Two lines of symmetry (vertical and horizontal), $180^\circ$ rotational symmetry (point symmetry)

2. [Flag of South Africa]
   - One line of symmetry (horizontal)

3. [Flag of Canada]
   - One line of symmetry (vertical)

4. [Flag of United Kingdom]
   - Two lines of symmetry (vertical and horizontal), $180^\circ$ rotational symmetry (point symmetry)

5. [Flag of Honduras]
   - One line of symmetry (vertical)

6. [Flag of Somalia]
   - One line of symmetry (vertical)

Tell whether each three-dimensional object has reflectional symmetry about a plane, rotational symmetry about a line, or both.

7. a pen both
8. a juice box reflectional symmetry in a plane
9. a candle both
10. a sofa reflectional symmetry in a plane
A dilation is a transformation in which a figure changes size. The preimage and image of a dilation are similar. The scale factor of the dilation is the same as the scale factor of these similar figures.

To find the scale factor, use the ratio of lengths of corresponding sides. If the scale factor of a dilation is greater than 1, the dilation is an enlargement. If it is less than 1, the dilation is a reduction.

Example

\( \triangle X'Y'Z' \) is the dilation image of \( \triangle XYZ \). The center of dilation is \( X \). The image of the center is itself, so \( X' = X \).

The scale factor, \( n \), is the ratio of the lengths of corresponding sides.

\[
 n = \frac{X'Z'}{XZ} = \frac{30}{12} = \frac{5}{2}
\]

This dilation is an enlargement with a scale factor of \( \frac{5}{2} \).

Exercises

For each of the dilations below, \( A \) is the center of dilation. Tell whether the dilation is a reduction or an enlargement. Then find the scale factor of the dilation.

1. \( A = A' \)  

   \[
   \frac{8}{10} = \frac{4}{5}
   \]

   enlargement; \( \frac{5}{4} \)

2. \( A = A' \)  

   \[
   \frac{8}{10} = \frac{4}{5}
   \]

   reduction; \( \frac{4}{5} \)

3. \( AB = 2; A'B' = 3 \)  

   enlargement; \( \frac{3}{2} \)

4. \( DE = 3; D'E' = 6 \)  

   enlargement; \( 2 \)

5. The image of a mosquito under a magnifying glass is six times the mosquito’s actual size and has a length of 3 cm. Find the actual length of the mosquito. 0.5 cm
Problem

Quadrilateral \(ABCD\) has vertices \(A(-2, 0), B(0, 2), C(2, 0),\) and \(D(0, -2)\).
Find the image of \(ABCD\) under the dilation centered at the origin with scale factor 2. Then graph \(ABCD\) and its image.

To find the image of the vertices of \(ABCD\), multiply the \(x\)-coordinates and \(y\)-coordinates by 2.

\[
\begin{align*}
A(-2, 0) & \rightarrow A'(\text{-}4, 0) & B(0, 2) & \rightarrow B'(0, 4) \\
C(2, 0) & \rightarrow C'(4, 0) & D(0, -2) & \rightarrow D'(0, -4)
\end{align*}
\]

Exercises

Use graph paper to complete Exercise 6.

6. a. Draw a quadrilateral in the coordinate plane. Check students’ work.

b. Draw the image of the quadrilateral under dilations centered at the origin with scale factors \(\frac{1}{2}, 2,\) and 4.

Graph the image of each figure under a dilation centered at the origin with the given scale factor. Check students’ graphs.
9-6 Reteaching
Compositions of Reflections

Two congruent figures in a plane can be mapped onto one another by a single reflection, compositions of reflections, or glide reflections.

Compositions of two reflections may be either translations or rotations.

If a figure is reflected across two parallel lines, it is a translation.

If a figure is reflected across intersecting lines, it is a rotation.

For both translations and rotations, the preimage and the image have the same orientation.

The arrow is reflected first across line ℓ and then across line m. Lines ℓ and m are parallel. These two reflections are equivalent to translation of the arrow downward.

The triangle is reflected first across line ℓ and then across line m. Lines ℓ and m intersect at point X. These two reflections are equivalent to a rotation. The center of rotation is X and the angle of rotation is twice the angle of intersection, in this case $2 \times 90^\circ$, or $180^\circ$.

A composition of a translation and a reflection is a glide reflection. For both reflections and glide reflections, the image and the preimage have opposite orientations.

$\triangle N'O'P'$ is the image of $\triangle NOP$, for a glide reflection where the translation is $(x, y) \rightarrow (x + 4, y - 1)$ and the line of reflection is $y = -1$. 
**Problem**

What transformation maps the figure $ABCD$ onto the figure $EFGH$ shown at the right?

The transformation is a glide reflection. It involves a translation, or glide, followed by a reflection in a line parallel to the translation vector.

**Exercises**

- Draw two pairs of parallel lines that intersect as shown at the right.
- Draw a nonregular quadrilateral in the center of the four lines.
- Use paper folding and tracing to reflect the figure and its images so that there is a figure in each of the nine sections.
- Label the figures 1 through 9 as shown.

Describe a transformation that maps each of the following.

1. figure 4 onto figure 6  translation
2. figure 1 onto figure 2   reflection
3. figure 7 onto figure 5  rotation
4. figure 2 onto figure 9  glide reflection
5. figure 1 onto figure 5  rotation
6. figure 6 onto figure 7  glide reflection
7. figure 8 onto figure 9  reflection
8. figure 2 onto figure 8  translation

$P(2, 3) \rightarrow P'$ by a glide reflection with the given translation and line of reflection. What are the coordinates of $P'$? (*Hint: it may help to graph the transformations.*)

9. $(x, y) \rightarrow (x + 3, y - 2); y = 0 \quad (5, -1)$
10. $(x, y) \rightarrow (x - 4, y + 2); x = 0 \quad (2, 5)$
11. $(x, y) \rightarrow (x, y - 3); y = x \quad (0, 2)$
12. $(x, y) \rightarrow (x - 2, y - 3); y = 4 \quad (0, 8)$
Tessellations are repeating patterns of figures that completely cover a plane without overlaps or gaps. A tessellation can be made by congruent copies of one figure or by congruent copies of multiple shapes.

If the figures in a tessellation are polygons, the sum of the measures of the angles around any vertex is $360^\circ$. To determine if a regular polygon will tessellate, find the measure of each angle of the polygon. If 360 is a multiple of this measure, the figure will tessellate.

A tessellation is shown at the right. The repeating figure in the tessellation is shown below.

**Problem**

Will a regular 24-gon tessellate a plane?

$\theta = \frac{180(n - 2)}{n}$

*Polygon Angle-Sum Theorem*

$\theta = \frac{180(24 - 2)}{24}$

Substitute 24 for $n$.

$\theta = 165$

Solve.

No multiple of 165 is equal to 360. A regular 24-gon will not tessellate.

**Exercises**

Find the repeating figure for each tessellation.

1. 

2. 


No; a regular 22-gon has angles that measure about $163.6^\circ$, which is not a factor of 360.
Tessellations have symmetry. The types of possible symmetry are reflectional (or line) symmetry, rotational symmetry, translational symmetry, and glide reflectional symmetry. A tessellation may have more than one type of symmetry.

**Problem**

What are the symmetries of the tessellation below?

The tessellation has line symmetry as shown by the dotted lines. It has rotational symmetry about the points shown. It has translational symmetry and glide reflection symmetry.

**Exercises**

Copy the figure at the right onto stiff paper or cardboard. Then cut it out.

4. Use the cutout to make a tessellation.

5. List the symmetries of the tessellation.

   line symmetry, rotational symmetry, translational symmetry, glide reflectional symmetry

List the symmetries of each tessellation.

6. line symmetry, rotational symmetry, translational symmetry, glide reflectional symmetry

7. rotational symmetry, translational symmetry

8. line symmetry, rotational symmetry, translational symmetry, glide reflectional symmetry
The area of a parallelogram is base $\times$ height. The base can be any side of the parallelogram. The height is the length of the corresponding altitude.

**Problem**

What is the area of $\square ABCD$?

$\overline{AB}$ is the correct base to use for the given altitude.

$$A = bh$$
$$A = 8(16) = 128 \text{ in.}^2$$

**Problem**

What is the value of $x$?

**Step 1:** Find the area of the parallelogram using the altitude perpendicular to $\overline{LM}$.

$$A = bh$$
$$A = 9(3) = 27 \text{ m}^2$$

**Step 2:** Use the area of the parallelogram to find the value of $x$.

$$A = bh$$
$$27 = 4.5x$$
$$x = 6 \text{ m}$$

**Exercises**

Find the area of each parallelogram.

1. $9 \text{ ft}$
   - $7 \text{ ft}$
   - $8 \text{ ft}$

   $63 \text{ ft}^2$

2. $3.5 \text{ mm}$
   - $13 \text{ mm}$
   - $12 \text{ mm}$

   $42 \text{ mm}^2$

3. $4 \text{ m}$
   - $10 \text{ m}$
   - $5.5 \text{ m}$

   $40 \text{ m}^2$

Find the value of $h$ for each parallelogram.

4. $7.5 \text{ yd}$
   - $12 \text{ yd}$
   - $6 \text{ yd}$

5. $16 \text{ m}$
   - $9 \text{ m}$
   - $2.25 \text{ m}$

6. $5 \text{ mi}$
   - $9.8 \text{ mi}$
   - $3.5 \text{ mi}$
The area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$. The base is any side of the triangle. The height is the length of the corresponding altitude.

**Problem**

A triangle has an area of 18 in.$^2$. The length of its base is 6 in. What is the corresponding height?

Draw a sketch. Then substitute into the area formula, and solve for $h$.

$$A = \frac{1}{2}bh$$

Substitute.

$$18 = \frac{1}{2}(6)h = 3h$$

Simplify.

$$h = 6 \text{ in.}$$

**Exercises**

7. Use graph paper. Draw an obtuse, an acute, and a right triangle, each with an area of 12 square units. Label the base and height of each triangle.

8. Draw a different obtuse, acute, and right triangle, each with an area of 12 square units. Label the base and height of each triangle. **Sample:**

9. A triangle has height 5 cm and base length 8 cm. Find its area. **20 cm$^2$**

10. A triangle has height 11 in. and base length 10 in. Find its area. **55 in.$^2$**

11. A triangle has area 24 m$^2$ and base length 8 m. Find its height. **6 m**

12. A triangle has area 16 ft$^2$ and height 4 ft. Find its base. **8 ft**

Find the area of each triangle.

13. **90 cm$^2$**

14. **56 in.$^2$**

15. **13.75 ft$^2$**

16. The figure at the right consists of a parallelogram and a triangle. What is the area of the figure? **1170 m$^2$**
10-2 Reteaching
Areas of Trapezoids, Rhombuses, and Kites

The area of a trapezoid is \( \frac{1}{2}h(b_1 + b_2) \), where \( h \) is the length of the height and \( b_1 \) and \( b_2 \) are the lengths of the two parallel bases.

**Problem**

What is the area of trapezoid \( WXYZ \)?

Draw an altitude to divide the trapezoid into a rectangle and a 30°-60°-90° triangle. In a 30°-60°-90° triangle, the length of the longer leg is \( \sqrt{3} \) times the length of the hypotenuse.

\[
h = \frac{\sqrt{3}}{2}(8) = 4\sqrt{3} \text{ in.}
\]

Use the formula for the area of a trapezoid.

\[
A = \frac{1}{2}h(b_1 + b_2)
\]

\[
= \frac{1}{2}(4\sqrt{3})(12 + 16)
\]

\[
= 56\sqrt{3} \text{ in.}^2
\]

The area of trapezoid \( WXYZ \) is \( 56\sqrt{3} \text{ in.}^2 \).

Find the area of each trapezoid. If necessary, leave your answer in simplest radical form.

1. \( 112 \text{ in.}^2 \)

2. \( 63 \text{ ft}^2 \)

3. \( 19.5 \text{ in.}^2 \)

4. \( 75 \text{ m}^2 \)

5. \( 16\sqrt{3} \text{ m}^2 \)

6. \( (240 - 50\sqrt{3}) \text{ cm}^2 \)

7. Find the area of a trapezoid with bases 9 m and 12 m and height 6 m. \( 63 \text{ m}^2 \)

8. Find the area of a trapezoid with bases 7 in. and 11 in. and height 3 in. \( 27 \text{ in.}^2 \)

9. Find the area of a trapezoid with bases 14 in. and 5 in. and height 11 in. \( 104.5 \text{ in.}^2 \)
The area of a rhombus or a kite is \( \frac{1}{2}d_1d_2 \), where \( d_1 \) and \( d_2 \) are the lengths of the diagonals. The diagonals bisect each other and intersect at right angles.

**Problem**

What is the area of rhombus \( ABCD \)?

First find the length of each diagonal.

\[
d_1 = 7 + 7 = 14 \quad \text{Diagonals of a rhombus bisect each other.}
\]

\[
8^2 = 7^2 + x^2 \quad \text{Pythagorean Theorem}
\]

\[
64 = 49 + x^2 \quad \text{Simplify.}
\]

\[
x = \sqrt{15}
\]

\[
d_2 = 2x = 2\sqrt{15} \text{ cm}
\]

Use the formula for the area of a rhombus.

\[
A = \frac{1}{2}d_1d_2 \quad \text{Substitute.}
\]

\[
= \frac{1}{2}(14)(2\sqrt{15}) \quad \text{Simplify.}
\]

\[
= 14\sqrt{15} \text{ cm}^2
\]

The area of rhombus \( ABCD \) is \( 14\sqrt{15} \text{ cm}^2 \).

Find the area of each rhombus. Leave your answer in simplest radical form.

10. \( 405 \text{ ft}^2 \)
11. \( 18 \sqrt{3} \text{ cm}^2 \)
12. \( 98 \sqrt{3} \text{ yd}^2 \)

Find the area of each kite. Leave your answer in simplest radical form.

13. \( 144 \text{ ft}^2 \)
14. \( 23.75 \sqrt{3} \text{ yd}^2 \)
15. \( (36 + 36\sqrt{3}) \text{ cm}^2 \)

16. Find the area of a kite with diagonals 13 cm and 14 cm. \( 91 \text{ cm}^2 \)

17. Find the area of a rhombus with diagonals 16 ft and 21 ft. \( 168 \text{ ft}^2 \)
If you circumscribe a circle around a regular polygon:
The radius of the circle is also the radius of the polygon, that is, the distance from the center to any vertex of the polygon.
The apothem is the perpendicular distance from the center to any side of the polygon.

**Problem**

What are $m\angle 1$, $m\angle 2$, and $m\angle 3$?

To find $m\angle 1$, divide the number of degrees in a circle by the number of sides.

$$m\angle 1 = \frac{360}{8} = 45$$

The apothem bisects the vertex angle, so you can find $m\angle 2$ using $m\angle 1$.

$$m\angle 2 = \frac{1}{2}m\angle 1 = \frac{1}{2}(45) = 22.5$$

Find $m\angle 3$ using the fact that $m\angle 3$ and $m\angle 2$ are complementary angles.

$$m\angle 3 = 90 - 22.5 = 67.5$$

For any regular polygon, the area is $A = \frac{1}{2} ap$. You can use the properties of special triangles to help you find the area of regular polygons.

**Problem**

What is the area of a regular quadrilateral (square) inscribed in a circle with radius 4 cm?

Draw one apothem to the base to form a $45^\circ$-$45^\circ$-$90^\circ$ triangle. Using the $45^\circ$-$45^\circ$-$90^\circ$ Triangle Theorem, find the length of the apothem.

$$a = \frac{4}{\sqrt{2}}$$

The hypotenuse = $\sqrt{2} \cdot$ leg in a $45^\circ$-$45^\circ$-$90^\circ$ triangle.

$$a = \frac{4\sqrt{2}}{\sqrt{2}} = 2\sqrt{2} \text{ cm}$$

Rationalize the denominator.

The apothem has the same length as the other leg, which is half as long as a side.

To find the square’s area, use the formula for the area of a regular polygon.

$$A = \frac{1}{2} ap$$

$$= \frac{1}{2}(2\sqrt{2})(16\sqrt{2}) \quad \text{Substitute } p = 4(4\sqrt{2}) = 16\sqrt{2}. \quad \text{ substitute }$$

$$= 32 \text{ cm}^2$$

The area of a square inscribed in a circle with radius 4 cm is 32 cm$^2$. 

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Exercises

Each regular polygon has radii and apothem as shown. Find the measure of each numbered angle.

1. 
2. 
3.

Find the area of each regular polygon with the given apothem $a$ and side length $s$.

4. pentagon, $a = 4.1$ m, $s = 6$ m
   - $61.5$ m$^2$
5. hexagon, $a = 10.4$ in., $s = 12$ in.
   - $374.4$ in.$^2$
6. 7-gon, $a = 8.1$ cm, $s = 7.8$ cm
   - $221.13$ cm$^2$
7. octagon, $a = 11.1$ ft, $s = 9.2$ ft
   - $408.48$ ft$^2$
8. nonagon, $a = 13.2$ in., $s = 9.6$ in.
   - $570.24$ in.$^2$
9. decagon, $a = 8.6$ m, $s = 5.6$ m
   - $240.8$ m$^2$

Find the area of each regular polygon. Round your answer to the nearest tenth.

10. 
11. 
12.

13. **Reasoning** How does the area of an equilateral triangle with sides 24 ft compare to the area of a regular hexagon with sides 24 ft? Explain.

The hexagon can be divided into 6 equilateral triangles with sides 24 ft, so it has 6 times the area of the equilateral triangle.

Find the area of each regular polygon with the given radius or apothem. If necessary, leave your answer in simplest radical form.

14. 
15. 
16.
Corresponding sides of similar figures are in proportion. The relationship between the lengths of corresponding sides in the two figures is called the scale factor. The perimeters and areas are related by the scale factor.

<table>
<thead>
<tr>
<th>Scale Factor</th>
<th>Ratio of Perimeters</th>
<th>Ratio of Areas</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{a}{b} )</td>
<td>( \frac{a}{b} )</td>
<td>( \frac{a^2}{b^2} )</td>
</tr>
</tbody>
</table>

**Problem**

The hexagons at the right are similar. What is the ratio (smaller to larger) of their perimeters and their areas?

The ratio of the corresponding sides is \( \frac{6}{12} \).

\[
\frac{P_{\text{smaller}}}{P_{\text{larger}}} = \frac{6}{12} = \frac{1}{2} \quad \text{Simplify.}
\]

The ratio of the areas is the square of the ratio of the corresponding sides.

\[
\frac{A_{\text{smaller}}}{A_{\text{larger}}} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}
\]

**Problem**

The rectangles at the right are similar. The area of the smaller rectangle is 72 in.\(^2\). What is the area of the larger rectangle?

The ratio of corresponding sides is \( \frac{a}{b} = \frac{6}{9} = \frac{2}{3} \).

Set up a proportion and solve:

\[
\frac{A_{\text{smaller}}}{A_{\text{larger}}} = \frac{a^2}{b^2}
\]

\[
\frac{72}{A_{\text{larger}}} = \frac{2^2}{3^2} \quad \text{Substitute.}
\]

\[
A_{\text{larger}} = 72 \times \frac{9}{4} = 162 \text{ in.}^2 \quad \text{Cross Products Property}
\]

**Exercises**

The figures in each pair are similar. Compare the first figure to the second. Give the ratio of the perimeters and the ratio of the areas.

1. \( 3 \text{ in.} \quad 1 : 2 \); \( 6 \text{ in.} \)

2. \( 4 \text{ cm} \); \( 7 \text{ cm} \)

3. \( 15 \text{ ft} \); \( 6 \text{ ft} \)
The figures in each pair are similar. The area of one figure is given. Find the area of the other figure to the nearest whole number.

4. Area of smaller pentagon = 112 m²
   Area of larger pentagon = 252 m²

5. Area of smaller rectangle = 78 in.²
   Area of larger rectangle = 312 in.²

6. Area of larger triangle = 75 cm²
   Area of smaller triangle = 12 cm²

7. Area of smaller octagon = 288 ft²
   Area of larger octagon = 450 ft²

The scale factor of two similar polygons is given. Find the ratio of their perimeters and the ratio of their areas.

8. \( \frac{4}{3} \); \( \frac{16}{9} \)
9. \( \frac{5}{8} \); \( \frac{25}{64} \)
10. \( \frac{3}{7} \); \( \frac{9}{49} \)
11. \( \frac{9}{2} \); \( \frac{9}{2}, \frac{81}{4} \)

12. The area of a regular nonagon is 34 m². What is the area of a regular nonagon with sides five times the sides of the smaller nonagon? 850 m²

13. A town is installing a sandbox in the park. The sandbox will be in the shape of a regular hexagon. On the plans for the sandbox, the sides are 4 in. and the area is about 42 in.². If the actual area of the sandbox will be 168 ft², what will be the length of one side of the sandbox? 8 ft

14. The longer sides of a parallelogram are 6 ft. The longer sides of a similar parallelogram are 15 ft. The area of the smaller parallelogram is 27 ft². What is the area of the larger parallelogram? 168.75 ft²

15. The shortest side of a pentagon is 4 cm. The shortest side of a similar pentagon is 9 cm. The area of the larger pentagon is 243 cm². What is the area of the smaller pentagon? 48 cm²

16. The scale factor of two similar floors is 5 : 6. It costs $340 to tile the smaller floor. At that rate, how much would it cost to tile the larger floor? $489.60
You can use the trigonometric ratios of sine, cosine, and tangent to help you find the area of regular figures.

\[ \sin \angle A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{CB}{AB} \]

\[ \cos \angle A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{AC}{AB} \]

\[ \tan \angle A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{CB}{AC} \]

**Problem**

What is the area of a regular hexagon with side 12 cm?

Area = \( \frac{1}{2}ap \), where \( a = \text{apothem} \)

\[ p = \text{perimeter} \]

\[ p = 6 \times 12 = 72 \text{ cm}, \text{ because the figure is 6-sided.} \]

Area = \( \frac{1}{2}a(72) = 36a \text{ cm}^2 \)

To find \( a \), examine \( \triangle AOB \) above. The apothem is measured along \( OM \), which divides \( \triangle AOB \) into congruent triangles.

\[ AM = \frac{1}{2}AB = 6 \]

\[ m\angle AOM = \frac{1}{2}m\angle AOB \]

\[ = \frac{1}{2} \left( \frac{360}{6} \right) \]

\[ = 30 \]

Divide 360 by 6 because there are six congruent central angles.

So, by trigonometry, \( \tan 30^\circ = \frac{AM}{a} \).

\[ \tan 30^\circ = \frac{6}{a} \]

\[ a = \frac{6}{\tan 30^\circ} \]

Finally, area = \( \frac{1}{2}ap = \frac{1}{2} \left( \frac{6}{\tan 30^\circ} \right)(72) \approx 374.1 \text{ cm}^2 \).

**Exercises**

Find the area of each regular polygon. Round your answers to the nearest tenth.

1. octagon with side 2 in. \( 19.3 \text{ in}^2 \)
2. decagon with side 4 cm \( 123.1 \text{ cm}^2 \)
3. pentagon with side 10 in. \( 172.0 \text{ in}^2 \)
4. 20-gon with side 40 in. \( 50,510.0 \text{ in}^2 \)
Theorem 10-8 Area of a Triangle Given SAS

\[ A = \frac{1}{2} bc \sin A \]

**Problem**

What is the area of the triangle? Round your answer to the nearest tenth.

\[ A = \frac{1}{2} bc \sin A \]

\[ = \frac{1}{2} (8)(11) \sin 62^\circ \]

\[ = 38.84969 \text{ cm}^2 \]

\[ = 38.8 \text{ cm}^2 \]

Round to the nearest tenth.

The area of the triangle is about 38.8 cm\(^2\).

**Exercises**

Find the area of each triangle. Round your answers to the nearest tenth.

5. \[ \text{Area} = 25.4 \text{ m}^2 \]

6. \[ \text{Area} = 53.5 \text{ in}^2 \]

7. \[ \text{Area} = 107.9 \text{ cm}^2 \]

8. \[ \text{Area} = 135.4 \text{ mm}^2 \]

9. \[ \text{Area} = 310.9 \text{ ft}^2 \]

10. \[ \text{Area} = 55.2 \text{ m}^2 \]

11. \( ABCDEF \) is a regular hexagon with center \( H \) and side 12 m. Find each measure. If necessary, round your answers to the nearest tenth.

- a. \( m\angle BHC \) 60
- b. \( m\angle BHG \) 30
- c. \( BG \) 6.0 m
- d. \( HG \) 10.4 m
- e. perimeter of \( ABCDEF \) 72.0 m
- f. area of \( ABCDEF \) 374.1 m\(^2\)
The circumference is the measure of the outside edge of a circle. Sections of the circumference are called arcs. There are three types of arcs.

<table>
<thead>
<tr>
<th>Minor arcs</th>
<th>Semicircles</th>
<th>Major arcs</th>
</tr>
</thead>
<tbody>
<tr>
<td>degrees</td>
<td>less than 180°</td>
<td>exactly 180°</td>
</tr>
<tr>
<td>name</td>
<td>named by two endpoints</td>
<td>named by three endpoints</td>
</tr>
<tr>
<td>example</td>
<td>(\overline{AB}) and (\overline{CB}) are minor arcs</td>
<td>(\overline{ABC}) is a semicircle</td>
</tr>
<tr>
<td>measure</td>
<td>measure = measure of corresponding central angle</td>
<td>measure = 180</td>
</tr>
</tbody>
</table>

The circumference of a circle is \(C = 2\pi r\), or \(C = \pi d\).

**Problem**

A circle has a 4.5-ft radius. What is the circumference of the circle?

\[
C = 2\pi r \\
C = 2(3.14)(4.5) \quad \text{Substitute 3.14 for } \pi \text{ and 4.5 for } r. \\
C = 28.26 \quad \text{Simplify.}
\]

The measure of an arc is in degrees. The arc’s length depends on the size of the circle because it represents a fraction of the circumference.

\[
\text{Length of } \overline{AB} = \frac{m\overline{AB}}{360} \cdot 2\pi r
\]

**Problem**

What is the length of the darkened arc? Leave your answer in terms of \(\pi\).

\[
\text{Length of } \overline{AB} = \frac{m\overline{AB}}{360} \cdot 2\pi r \\
= \frac{300}{360} \cdot 2\pi(15) \quad \text{Substitute 300 for } m\overline{AB} \text{ and 15 for } r. \\
= 25\pi \text{ m} \quad \text{Simplify.}
\]

The length of the darkened arc is \(25\pi\) m.
Exercises

Name the following in \( \bigcirc P \).

1. the minor arcs
   - Answers may vary. Sample: \( RQ; RU; RS; QU; QT; UT; US; TS \)
2. the major arcs
   - Answers may vary. Sample: \( RUS; RSU; RUQ; QSU; QST; UQS; UQT; TQS \)
3. the semicircles \( RUT; RST; SUQ; SRQ \)

Find the measure of each arc in \( \bigcirc A \).

4. \( WX \) 90
5. \( XY \) 77
6. \( WY \) 167
7. \( WXZ \) 270
8. \( YZW \) 193
9. \( XZY \) 283
10. \( ZWX \) 180
11. \( WZ \) 90
12. \( WYX \) 270

Find the circumference of each circle. Leave your answers in terms of \( \pi \).

13. \( 6\pi \) ft
14. \( 9\pi \) in.
15. \( 7\pi \) m

Find the length of each arc. Leave your answers in terms of \( \pi \).

16. \( SV \) \( \frac{5\pi}{6} \) cm
17. \( UV \) \( \frac{10\pi}{3} \) cm
18. \( SUT \) \( 5\pi \) cm
19. \( UTV \) \( \frac{20\pi}{3} \) cm
20. \( UT \) \( \frac{5\pi}{2} \) cm
21. \( VT \) \( \frac{25\pi}{6} \) cm

22. The wheel of a car is shown at the right. How far does the hubcap of the tire travel in one complete rotation? How far does the tire itself travel in one complete rotation? \( 18.8 \) in.; \( 69.1 \) in.

23. How far does the tip of a minute hand on a clock travel in 20 minutes if the distance from the center to the tip is 9 cm? Leave your answer in terms of \( \pi \). Then, round your answer to the nearest tenth. \( 6\pi \) cm; \( 18.8 \) cm
10-7  Reteaching
Areas of Circles and Sectors

Finding the Area of a Circle

The area of a circle is $A = \pi r^2$. So, to find the area of a circle, you need to know the radius, $r$. Sometimes you are given the radius directly. Sometimes you are given the diameter and have to divide by 2 to find the radius.

**Problem**

What is the area of $\odot S$?

- $r = \frac{6}{2} = 3$  \hspace{1cm} \text{Find the radius.}
- $A = \pi (3)^2$  \hspace{1cm} \text{Substitute.}
- $A = 9\pi$  \hspace{1cm} \text{Simplify.}
- $A \approx 28.3$  \hspace{1cm} \text{Approximate.}

The area of $\odot S$ is about 28.3 units$^2$.

Finding the Area of a Sector

To find the area of a sector, find the area of the circle, then multiply by the fraction of the circumference covered by the arc of the sector.

**Problem**

$\odot B$ has a radius of 2 and $m\widehat{AC} = 60$. What is the area of sector $ABC$?

First, find the area of $\odot B : A = (2)^2 \cdot \pi = 4\pi$

Then, find the fraction of the circumference covered by the arc.

$m\widehat{AC} \over 360 = \frac{60}{360} = \frac{6}{36} = \frac{1}{6}$

Last, multiply by the fraction.

Area of sector $ABC = \frac{1}{6} \cdot 4\pi = \frac{4}{6}\pi = \frac{2}{3}\pi \approx 2.1$

**Exercises**

Find the areas of circles and sectors described below. Write your answer in terms of $\pi$ and round to the nearest tenth.

1. circle with radius 5 \hspace{1cm} $25\pi \approx 78.5$
2. circle with diameter 16 \hspace{1cm} $64\pi \approx 201.1$
3. sector $ABC$ in $\odot B$ of radius 4 and $m\widehat{AC} = 90$ \hspace{1cm} $4\pi \approx 12.6$
4. sector $RST$ in $\odot S$ of diameter 12 and $m\widehat{RT} = 45$ \hspace{1cm} $\frac{9}{2}\pi \approx 14.1$
Finding the Area of a Segment

To find the area of a segment, subtract the area of the triangle from the area of the sector. So, use your knowledge of how to find the area of a sector.

**Problem**

What is the area of the shaded segment?

a. Find the area of sector \(ABC\).
   
   Circle: \(A = (9)^2 \cdot \pi = 81\pi\)
   
   Fraction: \(\frac{mAC}{360} = \frac{120}{360} = \frac{1}{3}\)
   
   Multiply: Area of sector \(ABC = \frac{1}{3} \cdot 81\pi = 27\pi\)

b. Find the area of \(\triangle ABC\).
   
   Triangle: \(A = \frac{1}{2}(AB)(BC) \sin \angle ABC\)
   
   \(A = \frac{1}{2}(9)^2 \sin 120 = \frac{81\sqrt{3}}{4}\)

c. Subtract to find the area of the segment.

   Segment: \(A = 27\pi - \frac{81\sqrt{3}}{4} \approx 49.7\)

The area of the shaded segment is about 49.7 units\(^2\).

**Exercises**

Find the area of each shaded segment.

5. \(\frac{8}{3}\pi - 4\sqrt{3} \approx 1.4\)

6. \(16\pi - 32 \approx 18.3\)

7. \(14\pi - 18\sin 140 \approx 32.4\)

Find the area of the shaded region. Leave your answer in terms of \(\pi\) and in simplest radical form.

8. \(126\pi + 36\sqrt{2}\)

9. \(189\pi + 81\)

10. \(160\pi + 64\sqrt{2}\)

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10-8  Reteaching
Geometric Probability

Problem
If a dart lands at random on the poster at the right, what is the probability that the dart will land inside one the polygons?

Find the sum of the areas of the polygons.

area of polygons = area of parallelogram + area of triangle

\[ = (12)(10) + \frac{1}{2} (10)(16) \]

\[ = 120 + 80 \]

\[ = 200 \text{ in.}^2 \]

Find the total area of the poster.

\[ A = (24)(36) = 864 \text{ in.}^2 \]

Calculate the probability.

\[ P(\text{polygon}) = \frac{\text{area of polygons}}{\text{total area}} \]

\[ = \frac{200}{864} \]

\[ \approx 23\% \]

Exercises

Complete each exercise.

1. Use a compass to draw a circle with radius 1 in. on an index card. Check students’ work. Sample shown:

2. Calculate the theoretical probability that if a tack is dropped on the card, its tip will land in the circle. \( \approx 21\% \)

3. Lift a tack 12 in. above the index card and drop it. Repeat this 25 times. Record how many times the tip of the tack lands in the circle. (Ignore the times that the tack bounces off the card.) Calculate the experimental probability:

\[ P = \frac{\text{number of times tip landed in circle}}{25} \]

Check students’ work.

4. How do the probabilities you found in Exercises 2 and 3 compare? Sample: They are approximately equal.

5. If you repeated the experiment 100 times, what would you expect the results to be? Sample: The ratio would tend to be close to 21%.

6. If a dart lands at random on the poster at the right, what is the probability that the dart will land in a circle? \( \approx 43\% \)
For Exercises 7–10 give your answer as a ratio and as a percent. For Exercises 7 and 8 use square $ABCD$ at the right.

7. Point $P$ in square $ABCD$ is chosen at random.
   Find the probability that $P$ is in square $AXYZ$. $\frac{4}{16} = \frac{1}{4} = 25\%$

8. Find the probability that $P$ is not in square $AXYZ$. $\frac{3}{4} = 75\%$

For Exercises 9 and 10 use rectangle $ABCD$ at the right.

9. Point $P$ in rectangle $ABCD$ is chosen at random.
   Find the probability that $P$ is in square $QRST$. $\frac{3}{10} = 30\%$

10. Find the probability that $P$ is not in square $QRST$. $\frac{7}{10} = 70\%$

Give your answer in terms of $\pi$, then as a percent.

11. Point $P$ in square $ABCD$ is chosen at random.
   Find the probability that $P$ is in not in $\bigcirc S$. $1 - \frac{\pi}{4} \approx 21.5\%$

12. Point $P$ in $\bigcirc S$ is chosen at random.
   Find the probability that $P$ is in not in square $ABCD$. $1 - \frac{2}{\pi} \approx 36.3\%$

Point $P$ in $\bigcirc S$ is chosen at random. Find the probability that $P$ is in sector $ABC$. Give your answer in terms of a ratio, then as a percent.

13. $\frac{\frac{1}{6} \cdot 36\pi}{36\pi} = \frac{1}{6} \approx 16.7\%$

14. $\frac{\frac{1}{3} \cdot 9\pi}{9\pi} = \frac{1}{3} \approx 33.3\%$

15. $\frac{\frac{1}{4} \cdot 16\pi}{16\pi} = \frac{1}{4} = 25\%$

16. The cycle of the light on George Street at the intersection of George Street and Main Street is 10 seconds green, 5 seconds yellow, and 60 seconds red. If you reach the intersection at a random time, what is the probability that the light is red? $80\%$
A **polyhedron** is a three-dimensional figure with **faces** that are polygons. Faces intersect at **edges**, and edges meet at **vertices**.

Faces, vertices, and edges are related by **Euler’s Formula**: \( F + V = E + 2 \).

For two dimensions, such as a representation of a polyhedron by a **net**, Euler’s Formula is \( F + V = E + 1 \). \((F \) is the number of regions formed by \( V \) vertices linked by \( E \) segments.\)

**Problem**

What does a net for the doorstop at the right look like? Label the net with its appropriate dimensions.

**Exercises**

Complete the following to verify Euler’s Formula.

1. On graph paper, draw three other nets for the polyhedron shown above. Let each unit of length represent \( \frac{1}{2} \) in. **Sample:**

2. Cut out each net, and use tape to form the solid figure. **Check students’ work.**

3. Count the number of vertices, faces, and edges of one of the figures. **6 vertices, 5 faces, 9 edges**

4. Verify that Euler’s Formula, \( F + V = E + 2 \), is true for this polyhedron. \( F + V = E + 2, 5 + 6 = 9 + 2, 11 = 11 \)

Draw a net for each three-dimensional figure. **Samples:**

5.

6.
A cross section is the intersection of a solid and a plane. Cross sections can be many different shapes, including polygons and circles.

The cross section of this solid and this plane is a rectangle.
This cross section is a horizontal plane.

To draw a cross section, visualize a plane intersecting one face at a time in parallel segments. Draw the parallel segments, then join their endpoints and shade the cross section.

Exercises

Draw and describe the cross section formed by intersecting the rectangular prism with the plane described.

7. a plane that contains the vertical line of symmetry
   Answers may vary. a rectangle or a square

8. a plane that contains the horizontal line of symmetry
   a rectangle

9. a plane that passes through the midpoint of the top left edge, the midpoint of the top front edge, and the midpoint of the left front edge
   an isosceles triangle

10. What is the cross section formed by a plane that contains a vertical line of symmetry for the figure at the right?
    Answers may vary. a hexagon or a rectangle

11. Visualization What is the cross section formed by a plane that is tilted and intersects the front, bottom, and right faces of a cube?
    a triangle
### 11-2 Reteaching

**Surface Areas of Cylinders and Prisms**

A *prism* is a polyhedron with two congruent parallel faces called *bases*. The non-base faces of a prism are *lateral faces*. The dimensions of a right prism can be used to calculate its lateral area and surface area.

The lateral area of a right prism is the product of the perimeter of the base and the height of the prism.

\[ \text{L.A.} = ph \]

The surface area of a prism is the sum of the lateral area and the areas of the bases of the prism.

\[ \text{S.A.} = \text{L.A.} + 2B \]

#### Problem

What is the lateral area of the regular hexagonal prism?

\[ \text{L.A.} = ph \]

\[ p = 6(4 \text{ in.}) = 24 \text{ in.} \]  
Calculate the perimeter.

\[ \text{L.A.} = 24 \text{ in.} \times 13 \text{ in.} \]  
Substitute.

\[ \text{L.A.} = 312 \text{ in.}^2 \]  
Multiply.

The lateral area is 312 in.$^2$.

#### Problem

What is the surface area of the prism?

\[ \text{S.A.} = \text{L.A.} + 2B \]

\[ p = 2(7 \text{ m} + 8 \text{ m}) \]  
Calculate the perimeter.

\[ p = 30 \text{ m} \]  
Simplify.

\[ \text{L.A.} = ph \]

\[ \text{L.A.} = 30 \text{ m} \times 30 \text{ m} \]  
Substitute.

\[ \text{L.A.} = 900 \text{ m}^2 \]  
Multiply.

\[ B = 8 \text{ m} \times 7 \text{ m} \]  
Find base area.

\[ B = 56 \text{ m}^2 \]  
Multiply.

\[ \text{S.A.} = \text{L.A.} + 2B \]

\[ \text{S.A.} = 900 \text{ m}^2 + 2 \times 56 \text{ m}^2 \]  
Substitute.

\[ \text{S.A.} = 1012 \text{ m}^2 \]  
Simplify.

The surface area of the prism is 1012 m$^2$. 

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A cylinder is like a prism, but with circular bases. For a right cylinder, the radius of the base and the height of the cylinder can be used to calculate its lateral area and surface area.

Lateral area is the product of the circumference of the base (\(2\pi r\)) and the height of the cylinder. Surface area is the sum of the lateral area and the areas of the bases (\(2\pi r^2\)).

\[
\text{L.A.} = 2\pi rh \quad \text{or} \quad \pi dh
\]
\[
\text{S.A.} = 2\pi rh + 2\pi r^2
\]

**Problem**

The diagram at the right shows a right cylinder. What are the lateral area and surface area of the cylinder?

\[
\text{L.A.} = 2\pi rh \quad \text{or} \quad \pi dh
\]
\[
\text{L.A.} = 2\pi \times 4 \text{ in.} \times 9 \text{ in.} \quad \text{Substitute for } r \text{ and } h.
\]
\[
\text{L.A.} = 72\pi \text{ in.}^2 \quad \text{Multiply.}
\]

The lateral area is 72\(\pi\) in.\(^2\).

\[
\text{S.A.} = 2\pi rh + 2\pi r^2
\]
\[
\text{S.A.} = 2\pi \times 4 \text{ in.} \times 9 \text{ in.} + 2\pi \times (4 \text{ in.})^2 \quad \text{Substitute for } r \text{ and } h.
\]
\[
\text{S.A.} = 72\pi \text{ in.}^2 + 32\pi \text{ in.}^2 \quad \text{Multiply.}
\]
\[
\text{S.A.} = 104\pi \text{ in.}^2 \quad \text{Add.}
\]

The surface area is 104\(\pi\) in.\(^2\).

**Exercises**

Find the lateral area and surface area of each figure. Round your answers to the nearest tenth, if necessary.

1. \[565.5 \text{ cm}^2; \ 722.6 \text{ cm}^2\]
2. \[576 \text{ in.}^2; \ 864 \text{ in.}^2\]
3. \[138.3 \text{ m}^2; \ 153.3 \text{ m}^2\]
4. A cylindrical carton of raisins with radius 4 cm is 25 cm tall. If all surfaces except the top are made of cardboard, how much cardboard is used to make the raisin carton? Round your answer to the nearest square centimeter.

\[679 \text{ cm}^2\]
A pyramid is a polyhedron in which the base is any polygon and the lateral faces are triangles that meet at the vertex. In a regular pyramid, the base is a regular polygon. The height is the measure of the altitude of a pyramid, and the slant height is the measure of the altitude of a lateral face. The dimensions of a regular pyramid can be used to calculate its lateral area (L.A.) and surface area (S.A.).

\[
\text{L.A.} = \frac{1}{2} p \ell, \quad \text{where } p \text{ is the perimeter of the base and } \ell \text{ is slant height of the pyramid.}
\]
\[
\text{S.A.} = \text{L.A.} + B, \quad \text{where } B \text{ is the area of the base.}
\]

**Problem**

What is the surface area of the square pyramid to the nearest tenth?

\[
\begin{align*}
\text{S.A.} &= \text{L.A.} + B \\
\text{L.A.} &= \frac{1}{2} p \ell \\
p &= 4(4 \text{ m}) = 16 \text{ m} \\
\ell^2 &= \sqrt{2^2 + 10^2} = \sqrt{104} \\
\ell &= 10.2 \\
\text{L.A.} &= \frac{1}{2} (16 \text{ m})(10.2) = 81.6 \text{ m}^2 \\
B &= (4 \text{ m})(4 \text{ m}) = 16 \text{ m}^2 \\
\text{S.A.} &= 81.6 \text{ m}^2 + 16 \text{ m}^2 = 97.6 \text{ m}^2
\end{align*}
\]

The surface area of the square pyramid is about 97.6 m².

**Exercises**

Use graph paper, scissors, and tape to complete the following.

1. Draw a net of a square pyramid on graph paper. **Sample:**

2. Cut it out, and tape it together. **Check students’ work.**

3. Measure its base length and slant height. **Sample: base = 3 cm, slant height = 4 cm**

4. Find the surface area of the pyramid. **Sample: 33 cm²**

In Exercises 5 and 6, round your answers to the nearest tenth, if necessary.

5. Find the surface area of a square pyramid with base length 16 cm and slant height 20 cm. *896 cm²*

6. Find the surface area of a square pyramid with base length 10 in. and height 15 in. *416.2 in.²*
A cone is like a pyramid, except that the base of a cone is a circle. The radius of the base and cylinder height can be used to calculate the lateral area and surface area of a right cone.

\[
\text{L.A.} = \pi r \ell, \text{ where } r \text{ is the radius of the base and } \ell \text{ is slant height of the cone.}
\]

\[
\text{S.A.} = \text{L.A.} + B, \text{ where } B \text{ is the area of the base } (B = \pi r^2).
\]

**Problem**

What is the surface area of a cone with slant height 18 cm and height 12 cm? Begin by drawing a sketch.

Use the Pythagorean Theorem to find \( r \), the radius of the base of the cone.

\[
r^2 + 12^2 = 18^2
\]

\[
r^2 + 144 = 324
\]

\[
r^2 = 180
\]

\[
r \approx 13.4
\]

Now substitute into the formula for the surface area of a cone.

\[
\text{S.A.} = \text{L.A.} + B = \pi r \ell + \pi r^2 = \pi (13.4)(18) + 180\pi 
\]

\[
\approx 1323.2
\]

The surface area of the cone is about 1323.2 cm\(^2\).

**In Exercises 7–10, round your answers to the nearest tenth, if necessary.**

7. Find the surface area of a cone with radius 5 m and slant height 15 m. 314.2 m\(^2\)

8. Find the surface area of a cone with radius 6 ft and height 11 ft. 349.3 ft\(^2\)

9. Find the surface area of a cone with radius 16 cm and slant height 20 cm. 1809.6 cm\(^2\)

10. Find the surface area of a cone with radius 10 in. and height 15 in. 880.5 in.\(^2\)
Problem
Which is greater: the volume of the cylinder or the volume of the prism?

Volume of the cylinder: 
\[ V = Bh \]
\[ = \pi r^2 \cdot h \]
\[ = \pi (3)^2 \cdot 12 \]
\[ = 339.3 \text{ in}^3 \]

Volume of the prism: 
\[ V = Bh \]
\[ = s^2 \cdot h \]
\[ = 6^2 \cdot 12 \]
\[ = 432 \text{ in}^3 \]

The volume of the prism is greater.

Exercises
Find the volume of each object.

1. the rectangular prism part of the milk container
\[ 490 \text{ cm}^3 \]

2. the cylindrical part of the measuring cup
\[ 77.175\pi \text{ or about } 242.5 \text{ cm}^3 \]

Find the volume of each of the following. Round your answers to the nearest tenth, if necessary.

3. a square prism with base length 7 m and height 15 m
\[ 735 \text{ m}^3 \]

4. a cylinder with radius 9 in. and height 10 in.
\[ 2544.7 \text{ in}^3 \]

5. a triangular prism with height 14 ft and a right triangle base with legs measuring 9 ft and 12 ft
\[ 756 \text{ ft}^3 \]

6. a cylinder with diameter 24 cm and height 5 cm
\[ 2261.9 \text{ cm}^3 \]
Problem

What is the volume of the triangular prism?

Sometimes the height of a triangular base in a triangular prism is not given. Use what you know about right triangles to find the missing value. Then calculate the volume as usual.

$$\text{hypotenuse} = 18 \text{ cm}$$  \hspace{1cm} \text{Given}
$$\text{short leg} = 9 \text{ cm}$$  \hspace{1cm} \text{30°-60°-90° triangle theorem}
$$\text{long leg} = 9\sqrt{3} \text{ cm}$$  \hspace{1cm} \text{30°-60°-90° triangle theorem}

Volume of prism: \( V = Bh \)
\[
V = \left(\frac{1}{2}\right)(9)(9\sqrt{3})(12)
\]
\[
V \approx 841.8 \text{ cm}^3
\]

The volume of the triangular prism is about 841.8 cm\(^3\).

Exercises

Find the volume of each prism. Round to the nearest tenth.

7. \(22.5 \text{ cm}^3\)

8. \(2833.8 \text{ in.}^3\)

9. \(1208.0 \text{ mm}^3\)

10. \(7.1 \text{ in.}^3\)

11. \(42 \text{ m}^3\)

12. \(265.0 \text{ in.}^3\)

Find the volume of each composite figure to the nearest tenth.

13. \(76 \text{ ft}^3\)

14. \(96.8 \text{ in.}^3\)

15. \(111.4 \text{ in.}^3\)
11-5 Reteaching
Volumes of Pyramids and Cones

Problem

What is the volume of the square pyramid?

Sometimes the height of a triangular face in a square pyramid is not given. Here the slant height and the lengths of the sides of the base are given. Use what you know about right triangles to find the missing value. Then calculate the volume as usual.

\[ 7^2 + x^2 = 25^2 \]
Use the Pythagorean Theorem.

\[ 49 + x^2 = 625 \]
Substitute.

\[ x^2 = 625 - 49 \]
Isolate the variable.

\[ x^2 = 576 \]
Simplify.

\[ x = 24 \text{ cm} \]
Find the square root of each side.

Volume of the pyramid:

\[ V = \frac{1}{3} Bh \]
Use the formula for volume of a pyramid.

\[ = \frac{1}{3}(14 \times 14)(24) \]
Substitute.

\[ = 1568 \text{ cm}^3 \]
Simplify.

The volume of the square pyramid is 1568 cm\(^3\).

Exercises

Find the volume of each pyramid. Round to the nearest whole number.

1. \[ 34,992 \text{ cm}^3 \]
2. \[ 10,240 \text{ in.}^3 \]
3. \[ 150 \text{ m}^3 \]
4. \[ 400 \text{ in.}^3 \]
5. \[ 4800 \text{ yd}^3 \]
6. \[ 48 \text{ cm}^3 \]
11-5 Reteaching (continued)
Volumes of Pyramids and Cones

Problem

What is the volume of the cone?

Find the height of the cone.

\[ 13^2 = h^2 + 5^2 \]
\[ 169 = h^2 + 25 \]
\[ h^2 = 144 \]
\[ h = 12 \]

Use the Pythagorean Theorem.
Substitute.
Simplify.
Take the square root of each side.

Find the volume of the cone.

\[ V = \frac{1}{3} \pi r^2 h \]
\[ = \frac{1}{3} \pi (5)^2 \cdot 12 \]
\[ = 100 \pi \]
\[ = 314.2 \]

The volume of the cone is about 314.2 cm\(^2\).

Exercises

7. From the figures shown below, choose the pyramid with volume closest to the volume of the cone at the right.  

A.

B.

C.

Find the volume of each figure. Round your answers to the nearest tenth.

8. 

9. 

10. 

11. 

301.6 cm\(^3\)

1005.3 m\(^3\)

18.8 ft\(^3\)

211.6 m\(^3\)
11-6 Reteaching
Surface Areas and Volumes of Spheres

Problem

What are the surface area and volume of the sphere?
Substitute $r = 5$ into each formula, and simplify.

\[
\begin{align*}
S.A. &= 4\pi r^2 \\
&= 4\pi(5)^2 \\
&= 100\pi \\
&\approx 314.2
\end{align*}
\]

\[
\begin{align*}
V &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3}\pi(5)^3 \\
&= \frac{500\pi}{3} \\
&\approx 523.6
\end{align*}
\]

The surface area of the sphere is about 314.2 in.$^2$. The volume of the sphere is about 523.6 in.$^3$.

Exercises

Use the figures at the right to guide you in completing the following.

1. Use a compass to draw two circles, each with radius 3 in.
   Cut out each circle.
   Check students’ work.

2. Fold one circle in half three successive times. Number the central angles 1 through 8.
   Check students’ work.

3. Cut out the sectors, and tape them together as shown.
   Check students’ work.

4. Take the other circle, fold it in half, and tape it to the rearranged circle so that they form a quadrant of a sphere.
   Check students’ work.

5. The area of one circle has covered one quadrant of a sphere. How many circles would cover the entire sphere?
   four circles

6. How is the radius of the sphere related to the radius of the circle?
   They are the same.

Find the volume and surface area of a sphere with the given radius or diameter.
Round your answers to the nearest tenth.

7. \[V = 523.6 \text{ in.}^3; \quad S.A. = 314.2 \text{ in.}^2\]
8. \[V = 113.1 \text{ cm}^3; \quad S.A. = 113.1 \text{ cm}^2\]
9. \[V = 7,238.2 \text{ m}^3; \quad S.A. = 1,809.6 \text{ m}^2\]
Find the volume and surface area of the sphere. Round to the nearest tenth.

10. S.A. = 2463.0 in.\(^2\); 
   \(V = 11,494.0 \text{ in.}^3\)  

11. S.A. = 6,157,521.6 m\(^2\); 
   \(V = 1,436,755,040.2 \text{ m}^3\)  

12. S.A. = 12.6 cm\(^2\); 
   \(V = 4.2 \text{ cm}^3\)

13. S.A. = 1256.6 m\(^2\); 
   \(V = 4188.8 \text{ m}^3\)

14. S.A. = 50.3 ft\(^2\); 
   \(V = 33.5 \text{ ft}^3\)

15. S.A. = 153.9 m\(^2\); 
   \(V = 179.6 \text{ m}^3\)

A sphere has the volume given. Find its surface area to the nearest whole number.

16. 1436.8 m\(^3\)  
   \(616 \text{ m}^2\)  

17. 808 cm\(^3\)  
   \(420 \text{ cm}^2\)  

18. 72 m\(^3\)  
   \(84 \text{ m}^2\)

Find the volume of each sphere with the given surface area. Round to the nearest whole number.

19. 435 yd\(^2\)  
   \(853 \text{ yd}^3\)  

20. 907 cm\(^2\)  
   \(2569 \text{ cm}^3\)  

21. 28 m\(^2\)  
   \(14 \text{ m}^3\)

22. **Visualization** The region enclosed by the semicircle at the right is revolved completely about the \(x\)-axis.
   a. Describe the solid of revolution that is formed. a sphere
   
   b. Find its volume in terms of \(\pi\). \(85\frac{1}{3}\pi \text{ units}^3\)
   
   c. Find its surface area in terms of \(\pi\). \(64\pi \text{ units}^2\)

23. The sphere at the right fits snugly inside a cube with 18 cm edges. What is the volume of the sphere? What is the surface area of the sphere? Leave your answers in terms of \(\pi\). \(972\pi \text{ cm}^3; 324\pi \text{ cm}^2\)
Reteaching
Areas and Volumes of Similar Solids

When two solids are similar, their corresponding dimensions are proportional.

Rectangular prisms A and B are similar because the ratio of their corresponding dimensions is \( \frac{2}{3} \).

- height: \( \frac{8 \text{ m}}{12 \text{ m}} = \frac{2}{3} \)
- length: \( \frac{2 \text{ m}}{3 \text{ m}} = \frac{2}{3} \)
- width: \( \frac{4 \text{ m}}{6 \text{ m}} = \frac{2}{3} \)

The ratio of the corresponding dimensions of similar solids is called the scale factor. All the linear dimensions (length, width, and height) of a solid must have the same scale factor for the solids to be similar.

### Areas and Volumes of Similar Solids

<table>
<thead>
<tr>
<th>Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>- The ratio of corresponding areas of similar solids is the square of the scale factor.</td>
<td>- The ratio of the volumes of similar solids is the cube of the scale factor.</td>
</tr>
<tr>
<td>- The ratio of the areas of prisms A and B is ( \frac{2^2}{3^2} ), or ( \frac{4}{9} ).</td>
<td>- The ratio of the volumes of prisms A and B is ( \frac{2^3}{3^3} ), or ( \frac{8}{27} ).</td>
</tr>
</tbody>
</table>

**Problem**

The pyramids shown are similar, and they have volumes of 216 in.\(^3\) and 125 in.\(^3\). The larger pyramid has surface area 250 in.\(^2\).

What is the ratio of their surface areas?

What is the surface area of the smaller pyramid?

By Theorem 11-12, if similar solids have similarity ratio \( a : b \), then the ratio of their volumes is \( a^3 : b^3 \).

So,

\[
\frac{a^3}{b^3} = \frac{216}{125}
\]

\[
\frac{a}{b} = \frac{6}{5}
\]

Take the cube root of both sides to get \( a : b \).

\[
\frac{a^2}{b^2} = \frac{36}{25}
\]

Square both sides to get \( a^2 : b^2 \).

Ratio of surface areas = \( \frac{36}{25} \)

If the larger pyramid has surface area 250 in.\(^2\), let the smaller pyramid have surface area \( x \).

Then,

\[
\frac{250}{x} = \frac{36}{25}
\]

\[
36x = 6250
\]

\[
x = 173.6 \text{ in.}^2
\]

The surface area of the smaller pyramid is about 173.6 in\(^2\).
Exercises
Find the scale factors.

1. Similar cylinders have volumes of $200\pi$ in.$^3$ and $25\pi$ in.$^3$. \(2 : 1\)

2. Similar cylinders have surface areas of $45\pi$ in.$^2$ and $20\pi$ in.$^2$. \(3 : 2\)

Are the two figures similar? If so, give the scale factor.

3. yes; \(1 : 2\)  

4. no

Each pair of figures is similar. Use the given information to find the scale factor of the smaller figure to the larger figure.

5. \(3 : 4\)  

6. \(4 : 9\)

Find the ratio of volumes.

7. Two cubes have sides of length 4 cm and 5 cm. \(64 : 125\)

8. Two cubes have surface areas of 64 in.$^2$ and 49 in.$^2$. \(512 : 343\)

The surface areas of two similar figures are given. The volume of the larger figure is given. Find the volume of the smaller figure.

9. S.A. = 16 cm$^2$ \(32\) cm$^3$  
   S.A. = 100 cm$^2$  
   \(V = 500\) cm$^3$

10. S.A. = 6 ft$^2$ \(10\) ft$^3$  

11. S.A. = 45 m$^2$ \(135\) m$^3$

The volumes of two similar figures are given. The surface area of the smaller figure is given. Find the surface area of the larger figure.

12. \(V = 12\) in.$^3$ \(48\) in.$^2$  
   \(V = 96\) in.$^3$  
   S.A. = 12 in.$^2$

13. \(V = 6\) cm$^3$ \(96\) cm$^2$  
   \(V = 384\) cm$^3$  
   S.A. = 6 cm$^2$

14. \(V = 40\) ft$^3$ \(45\) ft$^2$  
   \(V = 135\) ft$^3$  
   S.A. = 20 ft$^2$
12-1 Reteaching
Tangent Lines

A tangent is a line that touches a circle at exactly one point. In the diagram, $AB$ is tangent to $\odot Q$. You can apply theorems about tangents to solve problems.

**Theorem 12-1**

If a line is tangent to a circle, then that line forms a right angle with the radius at the point where the line touches the circle.

**Theorem 12-2**

If a line in the same plane as a circle is perpendicular to a radius at its endpoint on the circle, then the line is tangent to the circle.

---

**Problem**

Use the diagram at the right to solve the problems below. $GH$ is tangent to $\odot K$.

What is the measure of $\angle G$?

Because $GH$ is tangent to $\odot K$, it forms a right angle with the radius. The sum of the angles of a triangle is always 180. Write an equation to find $m\angle G$.

\[
m\angle G + m\angle H + m\angle K = 180
\]

\[
m\angle G + 90 + 68 = 180
\]

\[
m\angle G + 158 = 180
\]

\[
m\angle G = 22
\]

So, the measure of $\angle G$ is 22 and the length of the radius is 3.5 units.

---

**Exercises**

In each circle, what is the value of $x$?

1. $O \quad x \quad 62$
2. $O \quad 59 \quad x \quad 31$
3. $O \quad x \quad 33 \quad 57$
12-1 Reteaching (continued)
Tangent Lines

In each circle, what is the value of $r$?

4. $\frac{3}{4}$
5. $\frac{9}{15}$
6. $\frac{8}{14}$

Theorem 12-3

If two segments are tangent to a circle from the same point outside the circle, then the two segments are equal in length.

In the diagram, $\overline{AB}$ and $\overline{BC}$ are both tangent to $\odot D$. So, they are also congruent.

When circles are drawn inside a polygon so that the sides of the polygon are tangents, the circle is inscribed in the figure. You can apply Theorem 12-3 to find the perimeter, or distance around the polygon.

**Problem**

$\odot M$ is inscribed in quadrilateral $ABCD$.

What is the perimeter of $ABCD$?

$ZA = AW = 9 \quad WB = BX = 5$

$CY = XC = 2 \quad YD = DZ = 3$

Now add to find the length of each side:

$AB = AW + WB = 9 + 5 = 14 \quad BC = BX + CX = 5 + 2 = 7$

$CD = CY + YD = 2 + 3 = 5 \quad DA = DZ + ZA = 3 + 9 = 12$

$14 + 7 + 5 + 12 = 38$

The perimeter is 38 in.

**Exercises**

Each polygon circumscribes a circle. What is the perimeter of each polygon?

7. $34\text{ cm}$
8. $35\text{ ft}$
9. $49\text{ cm}$
12-2 Reteaching
Chords and Arcs

Several relationships between chords, arcs, and the central angles of a circle are listed below. The converses of these theorems are also true.

**Theorem 12-4** Congruent central angles have congruent arcs.

**Theorem 12-5** Congruent central angles have congruent chords.

**Theorem 12-6** Congruent chords have congruent arcs.

**Theorem 12-7** Chords equidistant from the center are congruent.

**Problem**

What is the value of \(x\)?

\[
\begin{align*}
EF &= FG = 3.2 & \text{Given} \\
\overline{AB} &= \overline{DC} & \text{Chords equidistant from the center of a circle are congruent.} \\
DC &= DG + GC & \text{Segment Addition Postulate} \\
AB &= x + GC & \text{Substitution} \\
DG &= GC = 3.5 & \text{Given} \\
x &= 3.5 + 3.5 = 7 & \text{Substitution}
\end{align*}
\]

The values of \(x\) is 7.

**Exercises**

In Exercises 1 and 2, the circles are congruent. What can you conclude? *Answers may vary.*

**Samples below:**

3.2 \(\overline{GI} \cong \overline{IJ}\) and \(\overline{GI} \cong \overline{IJ}\)

24 \(\overline{WX} \cong \overline{ZY}\) and \(\overline{WX} \cong \overline{ZY}\)

Find the value of \(x\).
Useful relationships between diameters, chords, and arcs are listed below. To bisect a figure means to divide it exactly in half.

**Theorem 12-8** In a circle, if a diameter is perpendicular to a chord, it bisects that chord and its arc.

**Theorem 12-9** In a circle, if a diameter bisects a chord that is not a diameter of the circle, it is perpendicular to that chord.

**Theorem 12-10** If a point is an equal distance from the endpoints of a line segment, then that point lies on the perpendicular bisector of the segment.

**Problem**

What is the value of $x$ to the nearest tenth?

In this problem, $x$ is the radius. To find its value draw radius $BD$, which becomes the hypotenuse of right $\triangle BDE$. Then use the Pythagorean Theorem to solve.

\[
ED = CE = 3 \quad \text{A diameter perpendicular to a chord bisects the chord.}
\]

\[
x^2 = 3^2 + 4^2 \quad \text{Use the Pythagorean Theorem.}
\]

\[
x^2 = 9 + 16 = 25 \quad \text{Solve for } x^2.
\]

\[
x = 5 \quad \text{Find the positive square root of each side.}
\]

The value of $x$ is 5.

**Exercises**

Find the value of $x$ to the nearest tenth.

6. \hspace{1cm} 9.8

7. \hspace{1cm} 15.9

8. \hspace{1cm} 4.9

Find the measure of each segment to the nearest tenth.

9. Find $c$ when $r = 6$ cm and $d = 1$ cm. **11.8 cm**

10. Find $c$ when $r = 9$ cm and $d = 8$ cm. **8.2 cm**

11. Find $d$ when $r = 10$ in. and $c = 10$ in. **8.7 in.**

12. Find $d$ when $r = 8$ in. and $c = 15$ in. **2.8 in.**
Two chords with a shared endpoint at the vertex of an angle form an inscribed angle. The intercepted arc is formed where the other ends of the chords intersect the circle.

In the diagram at the right, chords $AB$ and $BC$ form inscribed $\angle ABC$. They also create intercepted arc $\overline{AC}$.

The following theorems and corollaries relate to inscribed angles and their intercepted arcs.

**Theorem 12-11:** The measure of an inscribed angle is half the measure of its intercepted arc.

- **Corollary 1:** If two inscribed angles intercept the same arc, the angles are congruent. So, $m \angle A \cong m \angle B$.
- **Corollary 2:** An angle that is inscribed in a semicircle is always a right angle. So, $m \angle W = m \angle Y = 90$.
- **Corollary 3:** When a quadrilateral is inscribed in a circle, the opposite angles are supplementary. So, $m \angle X + m \angle Z = 180$.

**Theorem 12-12:** The measure of an angle formed by a tangent and a chord is half the measure of its intercepted arc.

**Problem**

Quadrilateral $ABCD$ is inscribed in $\odot J$.

$m \angle ADC = 68$; $\overline{CE}$ is tangent to $\odot J$.

What is $m \angle ABC$? What is $m \overline{CB}$? What is $m \angle DCE$?

\[
\begin{align*}
  m \angle ABC + m \angle ADC &= 180 & \text{Corollary 3 of Theorem 12-11} \\
  m \angle ABC + 68 &= 180 & \text{Substitution} \\
  m \angle ABC &= 112 & \text{Subtraction Property} \\
  m \overline{DB} &= m \overline{DC} + m \overline{CB} & \text{Arc Addition Postulate} \\
  180 &= 110 + m \overline{CB} & \text{Substitution} \\
  70 &= m \overline{CB} & \text{Simplify.} \\
  m \overline{CD} &= 110 & \text{Given} \\
  m \angle DCE &= \frac{1}{2} m \overline{CD} & \text{Theorem 12-12} \\
  m \angle DCE &= \frac{1}{2}(110) & \text{Substitution} \\
  m \angle DCE &= 55 & \text{Simplify.}
\end{align*}
\]

So, $m \angle ABC = 112$, $m \overline{CB} = 70$, and $m \angle DCE = 55$. 
Exercises

In Exercises 1–9, find the value of each variable.

1. \[ a = \frac{24}{48} \]

2. \[ \angle = 88^\circ \]

3. \[ \angle = \frac{90}{96} \]

4. \[ \angle = \frac{94}{56} \]

5. \[ \angle = \frac{76}{98} \]

6. \[ \angle = \frac{40}{200} \]

7. \[ \angle = \frac{80}{140} \]

8. \[ \angle = \frac{74}{90} \]

9. \[ \angle = \frac{112}{90} \]

Find the value of each variable. Lines that appear to be tangent are tangent.

10. \[ \angle = 106^\circ \]

11. \[ \angle = 112^\circ \]

12. \[ \angle = 108^\circ \]

Points A, B, and D lie on \( \odot C \). \( m\angle ACB = 40 \). \( m\overline{AB} < m\overline{AD} \). Find each measure.

13. \( m\angle AB \) 40

14. \( m\angle ADB \) 20

15. \( m\angle BAC \) 70

16. A student inscribes a triangle inside a circle. The triangle divides the circle into arcs with the following measures: 46°, 102°, and 212°. What are the measures of the angles of the triangle? \( 23; 51; 106 \)

17. A student inscribes \( NOPQ \) inside \( \odot Y \). The measure of \( m\angle N = 68 \) and \( m\angle Q = 94 \). Find the measures of the other angles of the quadrilateral. \( m\angle P = 112; m\angle Q = 86 \)
12-4  Reteaching
Angle Measures and Segment Lengths

**Problem**

In the circle shown, \( m\overline{BC} = 15 \) and \( m\overline{DE} = 35 \).

What are \( m \angle A \) and \( m \angle BFC \)?

Because \( \overline{AD} \) and \( \overline{AE} \) are secants, \( m \angle A \) can be found using Theorem 12-14.

\[
m \angle A = \frac{1}{2}(m\overline{DE} - m\overline{BC})
\]
\[
= \frac{1}{2}(35 - 15)
\]
\[
= 10
\]

Because \( \overline{BE} \) and \( \overline{CD} \) are chords, \( m \angle BFC \) can be found using Theorem 12-13.

\[
m \angle BFC = \frac{1}{2}(m\overline{DE} + m\overline{BC})
\]
\[
= \frac{1}{2}(35 + 15)
\]
\[
= 25
\]

So, \( m \angle A = 10 \) and \( m \angle BFC = 25 \).

**Exercises**

**Algebra**  Find the value of each variable.

1. \( 70^\circ \), \( 116^\circ \)
2. \( 20^\circ \), \( 28^\circ \)
3. \( 139^\circ \)
4. \( 35 \)
5. \( 90^\circ \)
6. \( 240^\circ \)
7. \( 36; 60; 48 \)
8. \( 64; 64; 52 \)
9. \( 46; 90; 44 \)
Segment Lengths

Here are some examples of different cases of Theorem 12-15.

A. Chords intersecting inside a circle:
   \[ \frac{part}{part} \cdot \frac{part}{part} = \frac{part}{part} \cdot \frac{part}{part} \]
   \[ 6x = 18 \]
   \[ x = \frac{18}{6} = 3 \]

B. Secants intersecting outside a circle:
   \[ \frac{outside}{whole} \cdot \frac{outside}{whole} = \frac{outside}{whole} \cdot \frac{whole}{whole} \]
   \[ x(x + 6) = 2(18 + 2) \]
   \[ x^2 + 6x = 40 \]
   \[ x^2 + 6x - 40 = 0 \]
   \[ (x + 10)(x - 4) = 0 \]
   \[ x = -10 \text{ or } x = 4 \]

C. Tangent and secant intersecting outside a circle:
   \[ \frac{tangent}{tangent} \cdot \frac{tangent}{tangent} = \frac{outside}{whole} \cdot \frac{whole}{whole} \]
   \[ x(x) = 4(4 + 5) \]
   \[ x^2 = 4(9) \]
   \[ x^2 = 36 \]
   \[ x = -6 \text{ or } x = 6 \]

Exercises

Algebra  Find the value of each missing variable.

10. \[ \begin{array}{c}
\quad \end{array} \]
11. \[ \begin{array}{c}
\quad \end{array} \]
12. \[ \begin{array}{c}
\quad \end{array} \]
13. \[ \begin{array}{c}
\quad \end{array} \]
14. \[ \begin{array}{c}
\quad \end{array} \]
15. \[ \begin{array}{c}
\quad \end{array} \]
Writing the Equation of a Circle from a Description

The standard equation for a circle with center \((h, k)\) and radius \(r\) is 
\[(x - h)^2 + (y - k)^2 = r^2\]. The opposite of the coordinates of the center appear in the equation. The radius is squared in the equation.

**Problem**

What is the standard equation of a circle with center \((-2, 3)\) that passes through the point \((-2, 6)\)?

**Step 1** Graph the points.

**Step 2** Find the radius using both given points. The radius is the distance from the center to a point on the circle, so \(r = 3\).

**Step 3** Use the radius and the coordinates of the center to write the equation.
\[
(x - (-2))^2 + (y - 3)^2 = 3^2
\]
\[
(x + 2)^2 + (y - 3)^2 = 9
\]

**Step 4** To check the equation, graph the circle. Check several points on the circle.
For \((1, 3)\): 
\[
(1 + 2)^2 + (3 - 3)^2 = 3^2 + 0^2 = 9
\]
For \((-5, 3)\): 
\[
(-5 + 2)^2 + (3 - 3)^2 = (-3)^2 + 0^2 = 9
\]
For \((-2, 0)\): 
\[
(-2 + 2)^2 + (0 - 3)^2 = 0^2 + (-3)^2 = 9
\]

The standard equation of this circle is \((x + 2)^2 + (y - 3)^2 = 9\).

**Exercises**

Write the standard equation of the circle with the given center that passes through the given point. Check the point using your equation.

1. center \((2, -4)\); point \((6, -4)\)
\[(x - 2)^2 + (y + 4)^2 = 16;
(6 - 2)^2 + (-4 + 4)^2 = 16\]

2. center \((0, 2)\); point \((3, -2)\)
\[x^2 + (y - 2)^2 = 25;
(3 - 0)^2 + (-2 - 2)^2 = 25\]

3. center \((-1, 3)\); point \((7, -3)\)
\[(x + 1)^2 + (y - 3)^2 = 100;
(7 + 1)^2 + (-3 - 3)^2 = 100\]

4. center \((1, 0)\); point \((0, 5)\)
\[(x - 1)^2 + y^2 = 26;
(0 - 1)^2 + 5^2 = 26\]

5. center \((-4, 1)\); point \((2, -2)\)
\[(x + 4)^2 + (y - 1)^2 = 45;
(2 + 4)^2 + (-2 - 1)^2 = 45\]

6. center \((8, -2)\); point \((1, 4)\)
\[(x - 8)^2 + (y + 2)^2 = 85;
(1 - 8)^2 + (4 + 2)^2 = 85\]
Writing the Equation of a Circle from a Graph

You can inspect a graph to find the coordinates of the circle’s center. Use the center and a point on the circle to find the radius. It is easier if you use a horizontal or vertical radius.

Problem

What is the standard equation of the circle in the diagram at the right?

Step 1 Write the coordinates of the center.

The center is at \(C(-5, 3)\).

Step 2 Find the radius. Choose a vertical radius: \(CZ\). The length is 6, so the radius is 6.

Step 3 Write the equation using the radius and the coordinates of the center.

\[
(x - h)^2 + (y - k)^2 = r^2
\]

\[
(x - (-5))^2 + (y - 3)^2 = 6^2
\]

\[
(x + 5)^2 + (y - 3)^2 = 36
\]

Step 4 Check two points on the circle.

For \((1, 3)\):

\[
(1 + 5)^2 + (3 - 3)^2 = 6^2 + 0^2 = 36
\]

For \((-11, 3)\):

\[
(-11 + 5)^2 + (3 - 3)^2 = 6^2 + 0^2 = 36
\]

The standard equation of this circle is \((x + 5)^2 + (y - 3)^2 = 36\).

Exercises

Write the standard equation of each circle. Check two points using your equation.

7.

\[
x^2 + (y - 4)^2 = 49
\]

Sample: \(0^2 + (-3 - 4)^2 = 49
\]

\[
7^2 + (4 - 4)^2 = 49
\]

8.

\[
(x + 2)^2 + (y + 4)^2 = 25
\]

Sample: \((3 + 2)^2 + (-4 + 4)^2 = 25
\]

\[
(-7 + 2)^2 + (-4 + 4)^2 = 25
\]
A locus is a set of points that all meet a condition or conditions. Finding a locus is a strategy that can be used to solve a word problem.

Problem

A family on vacation wants to hike on Oak Mountain and fish at North Pond and along the White River. Where on the river should they fish to be equidistant from North Pond and Oak Mountain?

Draw a line segment joining North Pond and Oak Mountain.

Construct the perpendicular bisector of that segment.

The family should fish where the perpendicular bisector meets the White River.

Exercises

Describe each of the following, and then compare your answers with those of a partner. Check students’ work.

1. the locus of points equidistant from your desk and your partner’s desk

2. the locus of points on the floor equidistant from the two side walls of your classroom

3. the locus of points equidistant from a window and the door of your classroom

4. the locus of points equidistant from the front and back walls of your classroom

5. the locus of points equidistant from the floor and the ceiling of your classroom

Use points \( A \) and \( B \) to complete the following.

6. Describe the locus of points in a plane equidistant from \( A \) and \( B \).
   the line perpendicular to \( \overline{AB} \) at its midpoint

7. How many points are equidistant from \( A \) and \( B \) and also lie on \( \overrightarrow{AB} \) ?
   Explain your reasoning.
   one, the midpoint of \( \overline{AB} \)

8. Describe the locus of points in space equidistant from \( A \) and \( B \).
   the plane perpendicular to \( \overline{AB} \) at its midpoint

9. Draw \( \overline{AB} \). Describe the locus of points in space 3 mm from \( \overline{AB} \).
   Check students’ drawings; a cylinder with radius 3 mm and central axis \( \overline{AB} \) and two hemispheres centered at \( A \) and \( B \) of radius 3 mm at the bases of the cylinder.
12-6  Reteaching (continued)

Locus: A Set of Points

10. Two students meet every Saturday afternoon to go running. Describe how they could use the map to find a variety of locations to meet that are equidistant from their homes.

They could draw a line segment with their homes as the endpoints, and then find its perpendicular bisector through the town. Everywhere the line passes through a street is a possible meeting place.

Use what you know about geometric figures to answer the following questions.

11. Sam tells Tony to meet him in the northeast section of town, 1 mi from the town’s center. Tony looks at his map of the town and picks up his cell phone to call Sam for more information. Why?

The locus of points Sam describes is roughly a quarter of a circle, with a radius of 1 mi.

12. How can city planners place the water sprinklers at the park so they are always an equal distance from the two main paths of the park?

The sprinklers can be placed along two lines that bisect the angles formed by the two paths.

13. An old pirate scratches the following note into a piece of wood:

“The treasure is 50 ft from a cedar tree and 75 ft from an oak.” Under what conditions would this give you one point to dig? two? none?

- when the trees are 125 ft apart
- when the trees are less than 125 ft apart
- when the trees are more than 125 ft apart

14. A ski resort has cut a wide path through mountain trees. Skiers will be coming down the hill, but the resort also needs to install the chairlift in the same space. What design allows skiers to ski down the hill with the maximum amount of space between them and the trees and the huge poles that support the chairlift?

The resort should install the chairlift to the far left or right of the open area so that skiers can ski down the center line, equidistant from the line of poles and the line of trees.

15. A telecommunications company is building a new cell phone tower and wants to cover three different villages. What location allows all three villages to get equal reception from the new tower?

The tower could be placed at the circumcenter of a triangle formed by the three villages.